

# THE INVERSE COMMUTANT LIFTING PROBLEM: CHARACTERIZATION OF ASSOCIATED REDHEFFER LINEAR-FRACTIONAL MAPS

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**ABSTRACT.** It is known that the set of all solutions of a commutant lifting and other interpolation problems admits a Redheffer linear-fractional parametrization. The method of unitary coupling identifies solutions of the lifting problem with minimal unitary extensions of a partially defined isometry constructed explicitly from the problem data. A special role is played by a particular unitary extension, called the *central or universal unitary extension*. The coefficient matrix for the Redheffer linear-fractional map has a simple expression in terms of the universal unitary extension. The universal unitary extension can be seen as a unitary coupling of four unitary operators (two bilateral shift operators together with two unitary operators coming from the problem data) which has special geometric structure. We use this special geometric structure to obtain an inverse theorem (Theorem 8.4 as well as Theorem 9.3) which characterizes the coefficient matrices for a Redheffer linear-fractional map arising in this way from a lifting problem. When expressed in terms of Hellinger-space functional models (Theorem 10.3), these results lead to generalizations of classical results of Arov and to characterizations of the coefficient matrix-measures of the lifting problem in terms of the density properties of the corresponding model spaces. The main tool is the formalism of unitary scattering systems developed in [18], [45].

## 1. INTRODUCTION

One of the seminal results in the development of operator theory and its applications over the past half century is the Commutant Lifting Theorem: *given contraction operators  $T'$ ,  $T''$  on respective Hilbert spaces  $\mathcal{H}'$ ,  $\mathcal{H}''$  with respective isometric dilations  $\mathcal{V}'$  and  $\mathcal{V}''$  on respective Hilbert spaces  $\mathcal{K}' \supset \mathcal{H}'$  and  $\mathcal{K}'' \supset \mathcal{H}''$  and given a contractive operator  $X: \mathcal{H}' \rightarrow \mathcal{H}''$  such that  $XT' = T''X$ , then there exists a operator  $Y: \mathcal{K}' \rightarrow \mathcal{K}''$  also with  $\|Y\| \leq 1$  such that  $Y\mathcal{V}' = \mathcal{V}''Y$  and  $XP_{\mathcal{H}'} = P_{\mathcal{H}''}Y$  (where  $P_{\mathcal{H}'}$  and  $P_{\mathcal{H}''}$  are the orthogonal projections of  $\mathcal{K}'$  to  $\mathcal{H}'$  and  $\mathcal{K}''$  to  $\mathcal{H}''$  respectively). It is well known that the general case can be reduced to the case where  $T' = \mathcal{U}'_+$  is an isometry on a Hilbert space  $\mathcal{K}'_+$  with unitary extension  $\mathcal{U}'$  on  $\mathcal{K}' \supset \mathcal{K}'_+$  and where  $T''$  is a coisometry on  $\mathcal{K}''_-$  with unitary lift  $\mathcal{U}''$  on  $\mathcal{K}'' \supset \mathcal{K}''_-$ . Then this normalized commutant lifting problem can be formalized as follows:*

**Problem 1.1 (Lifting Problem).** *Given two unitary operators  $\mathcal{U}'$  and  $\mathcal{U}''$  on Hilbert spaces  $\mathcal{K}'$  and  $\mathcal{K}''$ , respectively, along with subspaces  $\mathcal{K}'_+ \subset \mathcal{K}'$  and  $\mathcal{K}''_- \subset \mathcal{K}''$*

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that are assumed to be  $*$ -cyclic for  $\mathcal{U}'$  and  $\mathcal{U}''$  respectively (i.e., the smallest reducing subspace for  $\mathcal{U}'$  containing  $\mathcal{K}'_+$  is the whole space  $\mathcal{K}'$  and similarly for  $\mathcal{U}''$  and  $\mathcal{K}''_-$ ) and such that

$$\mathcal{U}'\mathcal{K}'_+ \subset \mathcal{K}'_+, \quad \mathcal{U}''^*\mathcal{K}''_- \subset \mathcal{K}''_-, \quad (1.1)$$

and given a contractive operator  $X: \mathcal{K}'_+ \rightarrow \mathcal{K}''_-$  which satisfies the intertwining condition

$$X\mathcal{U}'|_{\mathcal{K}'_+} = P_{\mathcal{K}''_-}\mathcal{U}''X, \quad (1.2)$$

characterize all contractive intertwiners  $Y$  of  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$  which lift  $X$  in the (Halmos) sense that

$$P_{\mathcal{K}''_-}Y|_{\mathcal{K}'_+} = X. \quad (1.3)$$

An important special case of this theorem was first proved by Sarason [64]; there he also explains the connections with classical Nevanlinna-Pick and Carathéodory-Fejér interpolation. Since the result was first formulated and proved in its full generality by Sz.-Nagy-Foias [54] (see also [55]), applications have been made to a variety of other contexts, including Nevanlinna-Pick interpolation for operator-valued functions and best approximation by analytic functions to a given  $L^\infty$ -function in  $L^\infty$ -norm (the Nehari problem)—we refer to the books [21] and [22] for an overview of all these developments. Moreover, the theorem has been generalized to still other contexts, e.g., to representations of nest algebras/time-varying systems [20, 22] as well as representations of more exotic Hardy algebras [24, 59, 15, 12, 71, 52, 50] with applications to more exotic Nevanlinna-Pick interpolation theorems [60, 61, 53, 62]. There has also appeared a weighted version [72, 17] as well as a relaxed version [49, 23, 25, 29] of the theorem leading to still other types of applications. There are now also results on linear-fractional parametrizations for the set of all solutions (see [3, 44] for the Nehari problem—see also [58, Chapter 5] for an overview, see [21, Chapter XIV] and [22, Theorem VI.5.1] and the references there for the standard formulation Problem 1.1 of the Lifting Problem, see [26, 29, 30] for the relaxed version of the lifting theorem); in the context of classical Nevanlinna-Pick interpolation, such parametrization results go back to the papers of Nevanlinna [56, 57].

The associated inverse problem asks for a characterization of which Redheffer linear-fractional coefficient-matrices arise in this way for some Lifting Problem. The inverse problem has been studied much less than the direct problem; there are only a few publications in this direction ([7, 40, 42]. We refer also to [32, 33, 66, 67]) for some special cases of the inverse Lifting Problem (Nehari problem and Nevanlinna-Pick/Carathéodory-Schur interpolation problem), and the quite recent work [31] on the inverse version of the relaxed commutant lifting problem.

Our contribution here is to further develop the ideas in [44, 45] to obtain new results on the inverse problem (Theorems 8.4, 9.3, 10.3) in terms of certain invariants associated with a Hellinger-space model for the Lifting Problem.

The starting point for our approach is the coupling method first introduced by Adamjan-Arov-Kreĭn [2] and developed further in [19, 4, 5, 34, 37, 51, 35, 36, 46, 43, 44, 45, 25, 49] (some of these in several-variable or relaxed contexts—see also [65] for a nice exposition). In this approach one identifies solutions of the Lifting Problem with minimal unitary extensions of an isometry constructed in a natural way from the problem data. We use here the term *isometry* (sometimes also called *semiunitary operator*) in the following technical sense: we are given a Hilbert space

$\mathcal{H}_0$  and subspaces  $\mathcal{D}$  and  $\mathcal{D}_*$  of  $\mathcal{H}_0$  together with a linear operator  $V$  which maps  $\mathcal{D}$  isometrically onto  $\mathcal{D}_*$ ; we then say that  $V$  is an *isometry on  $\mathcal{H}_0$  with domain  $\mathcal{D}$  and codomain  $\mathcal{D}_*$* . By a *minimal unitary extension* of  $V$  we mean a unitary operator  $\mathcal{U}$  on a Hilbert space  $\mathcal{K}$  containing  $\mathcal{H}_0$  as a subspace such that the restriction of  $\mathcal{U}$  to  $\mathcal{D}$  agrees with  $V$  and the smallest  $\mathcal{U}$ -reducing subspace containing  $\mathcal{H}_0$  is all of  $\mathcal{K}$ . From the work of Arov-Grossman [8, 9] and Katsnelson-Kheifets-Yuditskii [34], it is known that there is a special unitary colligation  $U_0$  (called the universal unitary colligation) so that any such unitary extension  $\mathcal{U}^*$  of  $V$  arises as the lower feedback connection  $\mathcal{U}^* = \mathcal{F}_\ell(U_0, U_1)$  of  $U_0$  with a free-parameter unitary colligation  $U_1$  (see Theorem 6.1). A special unitary extension of  $V$  is obtained as the unitary dilation  $\mathcal{U}_0^*$  of the universal unitary colligation  $U_0$  (or, in the language of [10],  $\mathcal{U}_0$  is the unitary evolution operator for the Lax-Phillips scattering system in which  $U_0$  is embedded). This special unitary extension  $\mathcal{U}_0^*$  of  $V$  is called the *universal unitary extension*.

Unlike other contexts where the “lurking isometry” approach has been used (see in particular [34, 46, 43, 47, 14]), the connection between unitary extensions  $\mathcal{U}^*$  of  $V$  and  $(\mathcal{U}', \mathcal{U}'')$ -intertwiners  $Y$  solving the Lifting Problem, as in [44, 45], involves an extra step: computation of the lift  $Y$  from the unitary extension  $\mathcal{U}$  is not immediately explicit but rather involves a wave-operator construction demanding computation of powers of  $\mathcal{U}$ . The lift  $Y$  is uniquely determined from its moments  $w_Y(n) = i_*^* Y^n i$  where  $i_*$  and  $i$  are certain isometric embedding operators (or *scale operators* in the sense of [18]). Calculation of such moments (the collection of which we call the *symbol* of the lift  $Y$ ) requires the computation of powers of  $\mathcal{U}^* = \mathcal{F}_\ell(U_0, U_1)$  in terms of the coefficients of the universal unitary colligation  $U_0$  determined by the problem data and the coefficients of the free-parameter unitary colligation  $U_1$  (or in terms of its characteristic function  $\omega(\zeta)$ ). In Section 2 we identify a general principle of independent interest for the explicit computation of the powers of an operator  $\mathcal{U}^*$  given as a feedback connection  $\mathcal{U}^* = \mathcal{F}_\ell(U_0, U_1)$  of two unitary colligations  $U_0$  and  $U_1$ . With the application of this general principle, we arrive at an explicit Redheffer-type linear-fractional parametrization of the set of symbols  $\{w_Y(n)\}_{n \in \mathbb{Z}}$  associated with the set of solutions  $Y$  of a Lifting Problem (see Theorem 7.1). The symbol for the Redheffer coefficient matrix is a simple explicit formula in terms of the universal unitary extension  $\mathcal{U}_0$  (see formula (8.40) in Theorem 8.6 below).

This general principle (already implicitly present in [44]) can be summarized as follows. Suppose that the operator  $\mathcal{U}$  is given as the lower feedback connection  $\mathcal{U}^* = \mathcal{F}_\ell(U_0, U_1)$  of two colligation matrices

$$U_0 = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{D} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{D}_* \end{bmatrix}, \quad U_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{D}_* \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{D} \end{bmatrix}.$$

(In our context we always have  $D_0 = 0$ ). Associated with any colligation matrix  $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{E} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{E}_* \end{bmatrix}$  is the discrete-time linear system

$$\Sigma_U : \begin{bmatrix} x(n+1) \\ e_*(n) \end{bmatrix} = U \begin{bmatrix} x(n) \\ e(n) \end{bmatrix}, \quad x(0) = x_0 \quad (1.4)$$

which recursively defines what we call the *augmented input-output map* (extending the usual input-output map in the sense that it takes into account an initial condition  $x(0) = x_0$  not necessarily equal to zero as well as the internal state trajectory

$\{x(n)\}_{n \in \mathbb{Z}}$ :

$$W(U)^+ := \begin{bmatrix} W(U)_0^+ & W(U)_2^+ \\ W(U)_1^+ & W(U_0)^+ \end{bmatrix} : \begin{bmatrix} x_0 \\ \{e(n)\}_{n \in \mathbb{Z}_+} \end{bmatrix} \mapsto \begin{bmatrix} \{x(n)\}_{n \in \mathbb{Z}_+} \\ \{e_*(n)\}_{n \in \mathbb{Z}_+} \end{bmatrix}$$

if  $\{e(n), x(n), e_*(n)\}_{n \in \mathbb{Z}_+}$  solves the system equations (1.4). Then the general principle asserts: *powers  $\mathcal{U}^n$  of  $\mathcal{U} = \mathcal{F}_\ell(U_0, U_1)$  can be computed via performing a feedback connection at the system-trajectory level:*

$$\left\{ \mathcal{U}^n \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \right\}_{n \in \mathbb{Z}_+} = \mathcal{F}_\ell(W(U_0)^+, W(U_1)^+) \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}.$$

The general structure for the universal unitary extension  $\mathcal{U}_0$  with embedded subspaces related to the original problem data  $(\mathcal{U}', \mathcal{K}')$ ,  $(\mathcal{U}'', \mathcal{K}'')$  and coefficient spaces  $\widehat{\Delta}$ ,  $\widehat{\Delta}_*$  for the free-parameter characteristic function can be viewed as a four-fold Adamjan-Arov (AA) unitary coupling in the general sense of [1]. In this setting one can identify the special geometry corresponding to the case where the four-fold AA-unitary coupling arises from a Lifting Problem. In this way we arrive at the inverse theorem (Theorems 8.4, 9.3, 10.3), specifically, a characterization of which Redheffer coefficient matrices arise as the coefficient matrix for the linear-fractional parametrization of the set of all solutions of some Lifting Problem (*with given operators  $\mathcal{U}', \mathcal{U}''$  and subspaces  $\mathcal{K}'_+ \subset \mathcal{K}', \mathcal{K}''_- \subset \mathcal{K}''$* ) generalizing results of [7, 40, 42] obtained in the context of the Nehari Problem and the bitangential Nevanlinna-Pick problem.

The solution of the inverse problem for the Lifting Problem as presented here appears to be quite different from the inverse problem considered in [31]. We discuss the connections between the results of this paper and those of [31] in detail in Remark 7.3 (for the direct problem) and in Remarks 9.5, 11.5 (for the inverse problem).

The paper is organized as follows. After the present Introduction, in Section 2 we present the general principle for computation of powers of  $\mathcal{U} = \mathcal{F}_\ell(U_0, U_1)$  via the trajectory-level feedback connection of the augmented input-output operator of  $U_0$  with that of  $U_1$ . Section 3 reviews preliminary material from [18] concerning Hellinger-space functional models for unitary operators equipped also with a scaling operator. Section 4 reviews basic ideas from [1] concerning the correspondence between contractive intertwiners  $Y$  of two unitary operators  $\mathcal{U}'$  and  $\mathcal{U}''$  on the one hand and unitary couplings  $\mathcal{U}$  of  $\mathcal{U}'$  and  $\mathcal{U}''$  on the other. Section 5 adds the constraint that the intertwiner  $Y$  should be a lift of a given contractive intertwiner  $X$  of restricted/compressed versions  $\mathcal{U}'_+$ ,  $\mathcal{U}''_-$  of  $\mathcal{U}'$ ,  $\mathcal{U}''$  and identifies the correspondence between solutions  $Y$  of the lifting problem and unitary extensions  $\mathcal{U}^*$  of the isometry  $V$  constructed directly from the data for the Lifting Problem. Section 6 recalls the result from [8, 9] that such unitary extensions arise as the lower feedback connection of the universal unitary colligation  $U_0$  with a free-parameter unitary colligation  $U_1$ . Section 7 uses the general principle from Section 2 to obtain a parametrization for the set of symbols  $\{w_Y(n)\}_{n \in \mathbb{Z}}$  associated with solutions  $Y$  of the Lifting Problem. Section 8 introduces the universal unitary extension. Here the universal unitary extension is identified as the four-fold AA-unitary coupling of the two unitary operators  $\mathcal{U}'$ ,  $\mathcal{U}''$  appearing in the Lifting-Problem data together with the bilateral shift operators associated with the input and output spaces for the free-parameter unitary colligation. Here the special geometric structure is identified which leads to the coordinate-free version of our inverse theorem (Theorem

8.4) characterizing which four-fold AA-unitary couplings arise in this way from a Lifting Problem. Here also is established the formula for the Redheffer-coefficient matrix in terms of the universal unitary extension. Sections 9 and 10 convert these results to more concrete function-theoretic form in the setting of Hellinger-model spaces. In particular, we get two more concrete versions of the inverse Theorem 8.4 (Theorems 9.3 and 10.3). In Section 11 we apply our results to the classical Nehari problem.

## 2. CALCULUS OF FEEDBACK CONNECTION OF UNITARY COLLIGATIONS

Suppose that we are given linear spaces  $\mathcal{X}_0, \tilde{\mathcal{X}}_0, \mathcal{X}_1, \tilde{\mathcal{X}}_1, \mathcal{F}, \mathcal{F}_*$  and linear operators presented in block matrix form

$$\begin{aligned} U_0 &= \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{F} \end{bmatrix} \rightarrow \begin{bmatrix} \tilde{\mathcal{X}}_0 \\ \mathcal{F}_* \end{bmatrix}, \\ U_1 &= \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{F}_* \end{bmatrix} \rightarrow \begin{bmatrix} \tilde{\mathcal{X}}_1 \\ \mathcal{F} \end{bmatrix}. \end{aligned} \quad (2.1)$$

We define the feedback connection  $U := \mathcal{F}_\ell(U_0, U_1) : \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{X}_1 \end{bmatrix} \rightarrow \begin{bmatrix} \tilde{\mathcal{X}}_0 \\ \tilde{\mathcal{X}}_1 \end{bmatrix}$  (when it exists) by

$$\begin{aligned} \mathcal{F}_\ell(U_0, U_1) \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} &= \begin{bmatrix} \tilde{x}_0 \\ \tilde{x}_1 \end{bmatrix} \text{ if there exist } f \in \mathcal{F} \text{ and } f_* \in \mathcal{F}_* \text{ so that} \\ \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \begin{bmatrix} x_0 \\ f \end{bmatrix} &= \begin{bmatrix} \tilde{x}_0 \\ f_* \end{bmatrix} \text{ and } \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} x_1 \\ f_* \end{bmatrix} = \begin{bmatrix} \tilde{x}_1 \\ f \end{bmatrix}. \end{aligned} \quad (2.2)$$

We also define the *elimination operator*  $\Gamma_\ell(U_0, U_1)$  (when it exists) by

$$\Gamma_\ell(U_0, U_1) : \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \mapsto \begin{bmatrix} f \\ f_* \end{bmatrix} \text{ if there exist } \tilde{x}_0, \tilde{x}_1 \text{ so that (2.2) holds.} \quad (2.3)$$

As explained in the following result, the feedback connection and elimination operator exist and are well-defined as long as the operator  $I - D_1 D_0$  is invertible as an operator on  $\mathcal{F}$ .

**Theorem 2.1.** *Suppose that we are given block-operator matrices  $U_0$  and  $U_1$  as in (2.1). Assume  $(I - D_1 D_0)^{-1}$  and hence also  $(I - D_0 D_1)^{-1}$  exist as operators on  $\mathcal{F}$  and  $\mathcal{F}_*$  respectively. Then the feedback connection (2.2) is well-posed, i.e., for each  $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \in \mathcal{X}_0 \oplus \mathcal{X}_1$  there exists a unique  $f \in \mathcal{F}$  and  $f_* \in \mathcal{F}_*$  so that the equations (2.2) determine a unique  $\begin{bmatrix} \tilde{x}_0 \\ \tilde{x}_1 \end{bmatrix} \in \tilde{\mathcal{X}}_0 \oplus \tilde{\mathcal{X}}_1$  which we then define to be  $\mathcal{F}_\ell(U_0, U_1)(\begin{bmatrix} x_0 \\ x_1 \end{bmatrix})$ .*

*More explicitly, the feedback connection operator  $\mathcal{F}_\ell(U_0, U_1) : \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \mapsto \begin{bmatrix} \tilde{x}_0 \\ \tilde{x}_1 \end{bmatrix}$  is given by*

$$\mathcal{F}_\ell(U_0, U_1) = \begin{bmatrix} A_0 + B_0(I - D_1 D_0)^{-1} D_1 C_0 & B_0(I - D_1 D_0)^{-1} C_1 \\ B_1(I - D_0 D_1)^{-1} C_0 & A_1 + B_1(I - D_0 D_1)^{-1} D_0 C_1 \end{bmatrix}. \quad (2.4)$$

*The elimination operator (2.3) which assigns instead the uniquely determined  $\begin{bmatrix} f \\ f_* \end{bmatrix}$  to  $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$  is then given explicitly by*

$$\Gamma_\ell(U_0, U_1) = \begin{bmatrix} (I - D_1 D_0)^{-1} D_1 C_0 & (I - D_1 D_0)^{-1} C_1 \\ (I - D_0 D_1)^{-1} C_0 & (I - D_0 D_1)^{-1} D_0 C_1 \end{bmatrix} : \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \mapsto \begin{bmatrix} f \\ f_* \end{bmatrix}. \quad (2.5)$$

*Proof.* The definition  $\mathcal{F}_\ell(U_0, U_1) \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} \tilde{x}_0 \\ \tilde{x}_1 \end{bmatrix}$  means that there is  $f \in \mathcal{F}$  and  $f_* \in \mathcal{F}_*$  so that (2.2) holds. From the second equation of the first system in (2.2) we have

$$f_* = C_0 x_0 + D_0 f.$$

Plug this into the second equation of the second system to get

$$\begin{aligned} f &= C_1 x_1 + D_1 f_* = C_1 x_1 + D_1 (C_0 x_0 + D_0 f) \\ &= D_1 C_0 x_0 + C_1 x_1 + D_1 D_0 f. \end{aligned}$$

Under the assumption that  $I - D_1 D_0$  is invertible we then can solve for  $f$  to get

$$f = (I - D_1 D_0)^{-1} D_1 C_0 x_0 + (I - D_1 D_0)^{-1} C_1 x_1.$$

Plug this into the second equation of the first system in (2.2) to then get

$$\begin{aligned} f_* &= C_0 x_0 + D_0 (I - D_1 D_0)^{-1} D_1 C_0 x_0 + D_0 (I - D_1 D_0)^{-1} C_0 x_1 \\ &= (I - D_0 D_1)^{-1} C_0 x_0 + (I - D_0 D_1)^{-1} D_0 C_1 x_1. \end{aligned}$$

In this way we get the formula (2.5) for the elimination operator  $\Gamma_\ell(U_0, U_1)$ . It is now a simple matter to plug in these values for  $f, f_*$  in terms of  $x_0, x_1$  into the first equations in the two systems (2.2) to arrive at the formula (2.4) for the feedback connection operator  $\mathcal{F}_\ell(U_0, U_1)$ .  $\square$

While the formula (2.4) exhibits  $\mathcal{F}_\ell(U_0, U_1)$  explicitly in terms of  $U_0$  and  $U_1$ , direct computation of powers  $U^n$  ( $n = 2, 3, \dots$ ) of  $U = \mathcal{F}_\ell(U_0, U_1)$  appears to be rather laborious. We next show how efficient computation of powers  $\mathcal{F}_\ell(U_0, U_1)$  can be achieved by use of a feedback connection at the level of system trajectories. Toward this end, we first introduce some useful notation.

For  $\mathcal{G}$  any linear space, we let  $\ell_{\mathcal{G}}(\mathbb{Z})$  (alternatively often written as  $\mathcal{G}^{\mathbb{Z}}$  in the literature) denote the space of all  $\mathcal{G}$ -valued functions on the integers  $\mathbb{Z}$ . Similarly we let  $\ell_{\mathcal{G}}(\mathbb{Z}_+)$  be the space of all  $\mathcal{G}$ -valued functions on the nonnegative integers  $\mathbb{Z}_+$ ; we often identify  $\ell_{\mathcal{G}}(\mathbb{Z}_+)$  with the subspace of  $\ell_{\mathcal{G}}(\mathbb{Z})$  consisting of all  $\mathcal{G}$ -valued functions on  $\mathbb{Z}$  which vanish on the negative integers. Similarly,  $\ell_{\mathcal{G}}(\mathbb{Z}_-)$  is the space of all  $\mathcal{G}$ -valued functions on  $\mathbb{Z}_-$  and is frequently identified with the subspace of  $\ell_{\mathcal{G}}(\mathbb{Z})$  consisting of all  $\mathcal{G}$ -valued functions on  $\mathbb{Z}$  vanishing on  $\mathbb{Z}_+$ . By  $\mathcal{P}^+$  and  $\mathcal{P}^-$  we denote the natural projections of  $\ell_{\mathcal{G}}(\mathbb{Z})$  onto  $\ell_{\mathcal{G}}(\mathbb{Z}_+)$  and  $\ell_{\mathcal{G}}(\mathbb{Z}_-)$ , respectively. Sometimes we will use notations

$$\vec{g}^+ = \mathcal{P}^+ \vec{g}, \quad \vec{g}^- = \mathcal{P}^- \vec{g}. \quad (2.6)$$

We also consider the bilateral shift operator  $J : \vec{g} \mapsto \vec{g}'$ , where  $\vec{g}'(n) = \vec{g}(n-1)$ .

Given a colligation matrix  $U$  of the form

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{E} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{E}_* \end{bmatrix} \quad (2.7)$$

we may consider the associated discrete-time input/state/output linear system

$$\begin{bmatrix} x(n+1) \\ e_*(n) \end{bmatrix} = U \begin{bmatrix} x(n) \\ e(n) \end{bmatrix} = \begin{bmatrix} Ax(n) + Be(n) \\ Cx(n) + De(n) \end{bmatrix}. \quad (2.8)$$

Given an initial state  $x(0) = x_0$  and an input string  $\vec{e} \in \ell_{\mathcal{E}}(\mathbb{Z}_+)$ , the system equations (2.8) recursively uniquely determine the state trajectory  $\vec{x} \in \ell_{\mathcal{X}}(\mathbb{Z}_+)$  and

the output string  $\vec{e}_* \in \ell_{\mathcal{E}_*}(\mathbb{Z}_+)$ ; explicitly we have

$$\begin{aligned} x(n) &= A^n x_0 + \sum_{k=0}^{n-1} A^{n-1-k} B e(k), \\ e_*(n) &= C A^n x_0 + \sum_{k=0}^{n-1} C A^{n-1-k} B e(k) + D e(n) \text{ for } n = 0, 1, 2, \dots \end{aligned} \quad (2.9)$$

If we view elements of  $\ell_{\mathcal{X}}(\mathbb{Z}_+)$  (and of  $\ell_{\mathcal{E}}(\mathbb{Z}_+)$  and  $\ell_{\mathcal{E}_*}(\mathbb{Z}_+)$ ) as column vectors, then operators between these various spaces can be represented as block matrices. We may then write the content of (2.9) in matrix form as

$$\begin{bmatrix} \vec{x} \\ \vec{e}_* \end{bmatrix} = \begin{bmatrix} W_0^+ & W_2^+ \\ W_1^+ & W^+ \end{bmatrix} \begin{bmatrix} x(0) \\ \vec{e} \end{bmatrix}$$

where the block-operator matrix

$$\mathbf{W}^+ := \begin{bmatrix} W_0^+ & W_2^+ \\ W_1^+ & W^+ \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \ell_{\mathcal{E}}(\mathbb{Z}_+) \end{bmatrix} \rightarrow \begin{bmatrix} \ell_{\mathcal{X}}(\mathbb{Z}_+) \\ \ell_{\mathcal{E}_*}(\mathbb{Z}_+) \end{bmatrix} \quad (2.10)$$

is given explicitly by

$$\begin{aligned} W_0^+ &= \begin{bmatrix} I_{\mathcal{X}} \\ A \\ \vdots \\ A^{n-1} \\ \vdots \end{bmatrix}, \quad W_2^+ = \begin{bmatrix} 0 & 0 & 0 & \dots \\ B & 0 & 0 & \dots \\ AB & B & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \\ A^{n-1}B & A^{n-2}B & A^{n-3}B & \dots & B & 0 & \dots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \\ W_1^+ &= \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \\ \vdots \end{bmatrix}, \quad W^+ = \begin{bmatrix} D & 0 & 0 \\ CB & D & 0 \\ CAB & CB & D \\ \vdots & \vdots & \ddots \\ CA^{n-1}B & CA^{n-2}B & CA^{n-3}B & \dots & D \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \end{aligned} \quad (2.11)$$

The operator  $W_0^+$  is the (forward-time) *initial-state/state-trajectory map*, the operator  $W_2^+$  is the *input/state-trajectory map*, the operator  $W_1^+$  is the *observation operator* and the operator  $W^+$  is what is traditionally known as the *input-output map* in the control literature. We note that the multiplication operator associated with  $W^+$  after applying the  $Z$ -transform

$$\vec{f} \mapsto \sum_{n \in \mathbb{Z}_+} \vec{f}(n) z^n$$

to the input and output strings  $\vec{e}$  and  $\vec{e}_*$  respectively has multiplier

$$\widehat{W}^+(z) = D + zC(I - zA)^{-1}B$$

equal to the *characteristic function* of the colligation  $U$  (also known as the *transfer function* of the linear system (2.8)). We shall refer to the whole  $2 \times 2$ -block operator matrix  $\mathbf{W}^+$  simply as the (forward-time) *augmented input/output map* associated with the colligation  $U$ .

If the colligation matrix  $U$  (2.7) is invertible, then we can also run the system in backwards time:

$$\begin{bmatrix} x(n) \\ e(n) \end{bmatrix} = U^{-1} \begin{bmatrix} x(n+1) \\ e_*(n) \end{bmatrix} = \begin{bmatrix} \alpha x(n+1) + \beta e_*(n) \\ \gamma x(n+1) + \delta e_*(n) \end{bmatrix} \quad (2.12)$$

where we set  $U^{-1} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} : \mathcal{X} \oplus \mathcal{E}_* \rightarrow \mathcal{X} \oplus \mathcal{E}$ . In this case, specification of an initial state  $x(0)$  and of the output string over negative time  $\vec{e}_* \in \ell_{\mathcal{E}_*}(\mathbb{Z}_-)$  determines recursively via the backward-time system equations (2.12) the state-trajectory over negative time  $\vec{x}_- \in \ell_{\mathcal{X}}(\mathbb{Z}_-)$  and the input string over negative time  $\vec{e}_- \in \ell_{\mathcal{E}}(\mathbb{Z}_-)$ . Explicitly we have

$$\begin{aligned} x(n) &= \alpha^n x(0) + \sum_{k=1}^n \alpha^{n-k} \beta e_*(-k), \\ e(-n) &= \gamma \alpha^{n-1} x(0) + \sum_{k=1}^{n-1} \gamma \alpha^{n-1-k} \beta e_*(-k) + \delta e_*(-n) \text{ for } n = 1, 2, \dots \end{aligned} \quad (2.13)$$

If we write elements  $\vec{x} = \{x(n)\}_{n \in \mathbb{Z}_-}$  of  $\ell_{\mathcal{X}}(\mathbb{Z}_-)$  as infinite column matrices

$$\vec{x} = \begin{bmatrix} \vdots \\ x(-3) \\ x(-2) \\ x(-1) \end{bmatrix}$$

then linear operators between spaces of the type  $\ell_{\mathcal{X}}(\mathbb{Z}_-)$  can be written as matrices with infinitely many rows as one ascends to the top. Then the relations (2.13) can be expressed in  $2 \times 2$ -block operator matrix form as

$$\begin{bmatrix} \vec{x}_- \\ \vec{e}_- \end{bmatrix} = \begin{bmatrix} W_0^- & W_1^- \\ W_2^- & W^- \end{bmatrix} \begin{bmatrix} x(0) \\ \vec{e}_* \end{bmatrix}, \quad (2.14)$$

where the  $2 \times 2$ -block operator matrix

$$\mathbf{W}^- := \begin{bmatrix} W_0^- & W_1^- \\ W_2^- & W^- \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \ell_{\mathcal{E}_*}(\mathbb{Z}_-) \end{bmatrix} \rightarrow \begin{bmatrix} \ell_{\mathcal{X}}(\mathbb{Z}_-) \\ \ell_{\mathcal{E}}(\mathbb{Z}_-) \end{bmatrix}$$

is given explicitly by

$$\begin{aligned} W_0^- &= \begin{bmatrix} \vdots \\ \alpha^n \\ \vdots \\ \alpha^2 \\ \alpha \end{bmatrix}, & W_1^- &= \begin{bmatrix} \ddots & & & & \\ & \beta & \dots & \alpha^{n-3}\beta & \alpha^{n-2}\beta & \alpha^{n-1}\beta \\ & & \ddots & \vdots & \vdots & \vdots \\ & & & \beta & \alpha\beta & \alpha^2\beta \\ & & & & \beta & \alpha\beta \\ & & & & & \beta \end{bmatrix}, \\ W_2^- &= \begin{bmatrix} \vdots \\ \gamma\alpha^n \\ \vdots \\ \gamma\alpha^2 \\ \gamma\alpha \end{bmatrix}, & W^- &= \begin{bmatrix} \ddots & & & & \\ & \delta & \dots & \gamma\alpha^{n-3}\beta & \gamma\alpha^{n-2}\beta & \gamma\alpha^{n-1}\beta \\ & & \ddots & \vdots & \vdots & \vdots \\ & & & \delta & \gamma\beta & \gamma\alpha\beta \\ & & & & \delta & \gamma\beta \\ & & & & & \delta \end{bmatrix}. \end{aligned} \quad (2.15)$$



Here the operators  $W_0^-$ ,  $W_1^-$ ,  $W_2^-$  and  $W^-$  are the *backward-time* versions of the *initial-state/state-trajectory*, *input/state-trajectory*, *observation* and *input/output* operators, respectively, and we refer to the aggregate operator  $\mathbf{W}^-$  simply as the *backward-time augmented input-output map*.

Let us now suppose that  $U_0$  and  $U_1$  are two colligation matrices

$$U_0 = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{D} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{D}_* \end{bmatrix}, \quad (2.16)$$

$$U_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{D}_* \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{D} \end{bmatrix} \quad (2.17)$$

such that  $I - D_1 D_0$  and hence also  $I - D_0 D_1$  are invertible on  $\mathcal{D}$  and on  $\mathcal{D}_*$  respectively. Then the feedback connection  $U = \mathcal{F}_\ell(U_0, U_1)$  is well-defined as an operator on  $\mathcal{X}_0 \oplus \mathcal{X}_1$  as explained in Theorem 2.1. Then we also have associated augmented input-output maps for  $U_0$  and  $U_1$  given by

$$\begin{aligned} \mathbf{W}(U_0)^+ &= \begin{bmatrix} W(U_0)_0^+ & W(U_0)_2^+ \\ W(U_0)_1^+ & W(U_0)^+ \end{bmatrix} : \begin{bmatrix} \mathcal{X}_0 \\ \ell_{\mathcal{D}}(\mathbb{Z}_+) \end{bmatrix} \rightarrow \begin{bmatrix} \ell_{\mathcal{X}_0}(\mathbb{Z}_+) \\ \ell_{\mathcal{D}_*}(\mathbb{Z}_+) \end{bmatrix} \\ \mathbf{W}(U_1)^+ &= \begin{bmatrix} W(U_1)_0^+ & W(U_1)_2^+ \\ W(U_1)_1^+ & W(U_1)^+ \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \ell_{\mathcal{D}_*}(\mathbb{Z}_+) \end{bmatrix} \rightarrow \begin{bmatrix} \ell_{\mathcal{X}_1}(\mathbb{Z}_+) \\ \ell_{\mathcal{D}}(\mathbb{Z}_+) \end{bmatrix}. \end{aligned} \quad (2.18)$$

Under the assumption that

$$I_{\ell_{\mathcal{D}}(\mathbb{Z}_+)} - W(U_1)^+ W(U_0)^+ \text{ is invertible on } \ell_{\mathcal{D}}(\mathbb{Z}_+), \quad (2.19)$$

it makes sense to form the feedback connection  $\mathcal{F}_\ell(\mathbf{W}(U_0)^+, \mathbf{W}(U_1)^+)$ . The following lemma guarantees that this connection is well-posed whenever the connection  $\mathcal{F}_\ell(U_0, U_1)$  is well-posed.

**Lemma 2.2.** *Let  $U_0$  and  $U_1$  be as in (2.16) and (2.17) and assume that  $I - D_1 D_0$  is invertible on  $\mathcal{D}$ . Then also  $I - W(U_1)^+ W(U_0)^+$  is invertible on  $\ell_{\mathcal{D}}(\mathbb{Z}_+)$ .*

*Proof.* From the formula for  $\mathbf{W}^+$  in (2.10) and (2.11), we see that  $W(U_1)^+$  and  $W(U_0)^+$  are given by lower triangular Toeplitz matrices with diagonal entries equal to  $D_1$  and  $D_0$  respectively. Hence  $I - W(U_1)^+ W(U_0)^+$  is also lower triangular Toeplitz with diagonal entry equal to  $I - D_1 D_0$ . A general fact is that an operator on  $\ell_{\mathcal{D}}(\mathbb{Z}_+)$  given by a lower triangular Toeplitz matrix with invertible diagonal entry is invertible on  $\ell_{\mathcal{D}}(\mathbb{Z}_+)$ . It follows that  $I - W(U_1)^+ W(U_0)^+$  is invertible on  $\ell_{\mathcal{D}}(\mathbb{Z}_+)$  as asserted.  $\square$

We now come to the main result of this section, namely: the computation of powers of  $\mathcal{F}_\ell(U_0, U_1)$  via the feedback connection  $\mathcal{F}_\ell(\mathbf{W}(U_0), \mathbf{W}(U_1))$ . For this purpose it is convenient to introduce the following general notation. For  $U$  an operator on a linear space  $\mathcal{K}$  and  $\mathcal{G}$  a subspace of  $\mathcal{K}$  with  $i_{\mathcal{G}}^*: \mathcal{K} \rightarrow \mathcal{G}$  the adjoint of the inclusion map  $i_{\mathcal{G}}: \mathcal{G} \rightarrow \mathcal{K}$ , we define an operator  $\Lambda_{\mathcal{G},+}(U): \mathcal{K} \rightarrow \ell_{\mathcal{G}}(\mathbb{Z}_+)$  (called the *Fourier representation operator*) by

$$\Lambda_{\mathcal{G},+}(U): k \rightarrow \{i_{\mathcal{G}}^* U^n k\}_{n \in \mathbb{Z}_+}. \quad (2.20)$$

Note that in case we take  $\mathcal{G} = \mathcal{K}$  we have simply

$$\Lambda_{\mathcal{K},+}(U): k \rightarrow \{U^n k\}_{n \in \mathbb{Z}_+}.$$

**Theorem 2.3.** *Suppose that we are given two colligation matrices (2.16), (2.17) such that  $I - D_1 D_0$  is invertible on  $\mathcal{D}$  and we set  $U = \mathcal{F}_\ell(U_0, U_1) \in \mathcal{L}(\mathcal{X}_0 \oplus$*

$\mathcal{X}_1$ ). Then the trajectory-level feedback connection operator  $\mathcal{F}_\ell(\mathbf{W}(U_0)^+, \mathbf{W}(U_1)^+)$  computes the powers of  $U = \mathcal{F}_\ell(U_0, U_1)$ :

$$\Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, +}(U) = \mathcal{F}_\ell(\mathbf{W}(U_0)^+, \mathbf{W}(U_1)^+): \mathcal{X}_0 \oplus \mathcal{X}_1 \rightarrow \ell_{\mathcal{X}_0 \oplus \mathcal{X}_1}(\mathbb{Z}_+). \quad (2.21)$$

Hence, after application of the natural identification between the spaces  $\ell_{\mathcal{X}_0 \oplus \mathcal{X}_1}(\mathbb{Z}_+)$  and  $\ell_{\mathcal{X}_0}(\mathbb{Z}_+) \oplus \ell_{\mathcal{X}_1}(\mathbb{Z}_+)$ , we have the explicit formulas

$$\Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, +}(U) = \begin{bmatrix} \Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, +}(U)_{11} & \Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, +}(U)_{12} \\ \Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, +}(U)_{21} & \Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, +}(U)_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{X}_1 \end{bmatrix} \rightarrow \begin{bmatrix} \ell_{\mathcal{X}_0}(\mathbb{Z}_+) \\ \ell_{\mathcal{X}_1}(\mathbb{Z}_+) \end{bmatrix}$$

where the matrix entries  $\Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, +}(U)_{ij}$  ( $i, j = 1, 2$ ) are given explicitly by

$$\begin{aligned} \Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, +}(U)_{11} &= W(U_0)_0^+ + W(U_0)_2^+(I - W(U_1)^+W(U_0)^+)^{-1}W(U_1)^+W(U_0)_1^+, \\ \Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, +}(U)_{12} &= W(U_0)_2^+(I - W(U_1)^+W(U_0)^+)W(U_1)_1^+, \\ \Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, +}(U)_{21} &= W(U_1)_2^+(I - W(U_0)^+W(U_1)^+)^{-1}W(U_0)_1^+, \\ \Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, +}(U)_{22} &= W(U_1)_0^+ + W(U_1)_2^+(I - W(U_0)^+W(U_1)^+)^{-1}W(U_0)^+W(U_1)_1^+. \end{aligned} \quad (2.22)$$

*Proof.* Note that Lemma 2.2 guarantees that the trajectory-level feedback connection  $\mathcal{F}_\ell(\mathbf{W}(U_0)^+, \mathbf{W}(U_1)^+)$  is well-posed. By definition, we see that

$$\mathcal{F}_\ell(\mathbf{W}(U_0)^+, \mathbf{W}(U_1)^+) \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} \{x_0(n)\}_{n \in \mathbb{Z}_+} \\ \{x_1(n)\}_{n \in \mathbb{Z}_+} \end{bmatrix} \quad (2.23)$$

means that

$$\begin{aligned} \begin{bmatrix} x_0(n+1) \\ d_*(n) \end{bmatrix} &= \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \begin{bmatrix} x_0(n) \\ d(n) \end{bmatrix}, \\ \begin{bmatrix} x_1(n+1) \\ d(n) \end{bmatrix} &= \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} x_1(n) \\ d_*(n) \end{bmatrix} \end{aligned} \quad (2.24)$$

for uniquely determined strings  $\{d(n)\}_{n \in \mathbb{Z}_+} \in \ell_{\mathcal{D}}(\mathbb{Z}_+)$  and  $\{d_*(n)\}_{n \in \mathbb{Z}_+} \in \ell_{\mathcal{D}_*}(\mathbb{Z}_+)$ . As  $U = \mathcal{F}_\ell(U_0, U_1)$ , the particular case  $n = 0$  of the equations (2.24) is just the assertion that

$$\begin{bmatrix} x_0(1) \\ x_1(1) \end{bmatrix} = U \begin{bmatrix} x_0(0) \\ x_1(0) \end{bmatrix}.$$

Inductively assume that

$$\begin{bmatrix} x_0(n) \\ x_1(n) \end{bmatrix} = U^n \begin{bmatrix} x_0(0) \\ x_1(0) \end{bmatrix}. \quad (2.25)$$

The  $n$ -th equation in (2.24) amounts to the assertion that

$$\begin{bmatrix} x_0(n+1) \\ x_1(n+1) \end{bmatrix} = U \begin{bmatrix} x_0(n) \\ x_1(n) \end{bmatrix}.$$

Combining with the inductive assumption (2.25) then gives us that (2.25) holds with  $n+1$  in place of  $n$  and hence (2.25) holds for all  $n = 0, 1, 2, \dots$ . Note next that (2.25) combined with (2.23) amounts to the identity (2.21). The explicit formulas (2.22) then follow from formula (2.4) with  $\mathbf{W}(U_0)^+, \mathbf{W}(U_1)^+$  as in (2.18) in place of  $U_0, U_1$ .  $\square$

If  $U_0$  and  $U_1$  are invertible with

$$U_0^{-1} = \begin{bmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{D}_* \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{D} \end{bmatrix}, \quad U_1^{-1} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{D} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{D}_* \end{bmatrix} \quad (2.26)$$

with  $I_{\mathcal{D}_*} - \gamma_1 \gamma_0$  invertible, a similar analysis can be brought to bear to compute negative powers of  $U = \mathcal{F}(U_0, U_1)$ . Let us introduce an operator  $\Lambda_{\mathcal{G}, -} : \mathcal{K} \rightarrow \ell_{\mathcal{G}}(\mathbb{Z}_-)$  (also called a *Fourier representation operator* along with (2.20)) by

$$\Lambda_{\mathcal{G}, -}(U) : k \mapsto \{i_{\mathcal{G}}^* U^n k\}_{n \in \mathbb{Z}_-}$$

where in particular

$$\Lambda_{\mathcal{K}, -}(U) k : \mapsto \{U^n k\}_{n \in \mathbb{Z}_-}.$$

Then we have the following backward-time result parallel to Theorem 2.3. As the proof is completely analogous, we omit the details of the proof.

**Theorem 2.4.** *Suppose that we are given two colligation matrices  $U_0$  and  $U_1$  as in (2.16), (2.17) with inverses as in (2.26) such that  $I - \delta_1 \delta_0$  is invertible on  $\mathcal{D}_*$  and we set  $U = \mathcal{F}_{\ell}(U_0, U_1) \in \mathcal{L}(\mathcal{X}_0 \oplus \mathcal{X}_1)$ . Then the trajectory-level feedback connection operator  $\mathcal{F}_{\ell}(\mathbf{W}(U_0)^-, \mathbf{W}(U_1)^-)$  computes the negative powers of  $U = \mathcal{F}_{\ell}(U_0, U_1)$ :*

$$\Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, -}(U) = \mathcal{F}_{\ell}(\mathbf{W}(U_0)^-, \mathbf{W}(U_1)^-) : \mathcal{X}_0 \oplus \mathcal{X}_1 \rightarrow \ell_{\mathcal{X}_0 \oplus \mathcal{X}_1}(\mathbb{Z}_-). \quad (2.27)$$

Hence, application of the natural identification between  $\ell_{\mathcal{X}_0 \oplus \mathcal{X}_1}(\mathbb{Z}_-)$  and  $\ell_{\mathcal{X}_0}(\mathbb{Z}_-) \oplus \ell_{\mathcal{X}_1}(\mathbb{Z}_-)$  leads to explicit formulas

$$\Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, -}(U) = \begin{bmatrix} \Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, -}(U)_{11} & \Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, -}(U)_{12} \\ \Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, -}(U)_{21} & \Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, -}(U)_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{X}_1 \end{bmatrix} \rightarrow \begin{bmatrix} \ell_{\mathcal{X}_0}(\mathbb{Z}_-) \\ \ell_{\mathcal{X}_1}(\mathbb{Z}_-) \end{bmatrix}$$

where the matrix entries  $\Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, +}(U)_{ij}$  ( $i, j = 1, 2$ ) are given explicitly by

$$\begin{aligned} \Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, -}(U)_{11} &= W(U_0)_0^- + W(U_0)_1^- (I - W(U_1)^- W(U_0)^-)^{-1} W(U_1)^- W(U_0)_2^-, \\ \Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, -}(U)_{12} &= W(U_0)_1^- (I - W(U_1)^- W(U_0)^-) W(U_1)_2^-, \\ \Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, -}(U)_{21} &= W(U_0)_1^- (I - W(U_0)^- W(U_1)^-)^{-1} W(U_0)_2^-, \\ \Lambda_{\mathcal{X}_0 \oplus \mathcal{X}_1, +}(U)_{22} &= W(U_1)_0^- + W(U_1)_1^- (I - W(U_0)^- W(U_1)^-)^{-1} W(U_0)^- W(U_1)_2^-. \end{aligned} \quad (2.28)$$

### 3. UNITARY SCATTERING SYSTEMS AND THEIR MODELS

**3.1. Unitary scattering systems.** Following [18] we define a *unitary scattering system* to be a collection  $\mathfrak{S}$  of the form

$$\mathfrak{S} = (\mathcal{U}, \Psi; \mathcal{K}, \mathcal{E}) \quad (3.1)$$

where  $\mathcal{K}$  (the *ambient space*) and  $\mathcal{E}$  (the *coefficient space*) are Hilbert spaces,  $\mathcal{U}$  is a unitary operator on  $\mathcal{K}$  (called the *evolution operator*), and  $\Psi$  (called the *scale operator*) is an operator from  $\mathcal{E}$  into  $\mathcal{K}$ . A fundamental object associated with any unitary scattering system  $\mathfrak{S}$  (3.1) is its so-called *characteristic function*  $w_{\mathfrak{S}}(\zeta)$  defined by

$$\begin{aligned} w_{\mathfrak{S}}(\zeta) &= \sum_{n=1}^{\infty} \Psi^* \mathcal{U}^n \Psi \bar{\zeta}^n + \sum_{n=0}^{\infty} \Psi^* \mathcal{U}^{*n} \Psi \zeta^n \\ &= \Psi^* [(I - \bar{\zeta} \mathcal{U})^{-1} + (I - \zeta \mathcal{U}^*)^{-1} - I] \Psi \\ &= (1 - |\zeta|^2) \Psi^* (I - \bar{\zeta} \mathcal{U})^{-1} (I - \zeta \mathcal{U}^*)^{-1} \Psi. \end{aligned} \quad (3.2)$$

From (3.2) we see that  $w_{\mathfrak{S}}(\zeta)$  is a positive harmonic operator-function (values are operators on  $\mathcal{E}$ )

$$w_{\mathfrak{S}}(\zeta) \geq 0 \text{ for } \zeta \in \mathbb{D} \text{ and } \frac{\partial^2}{\partial \bar{\zeta} \partial \zeta} w_{\mathfrak{S}}(\zeta) = 0 \text{ for all } \zeta \in \mathbb{D}. \quad (3.3)$$

If we introduce the convention

$$\zeta^{[n]} = \begin{cases} \zeta^n & \text{if } n \geq 0 \\ \bar{\zeta}^{-n} & \text{if } n < 0 \end{cases}$$

then the first formula in (3.2) can be written more succinctly as

$$w_{\mathfrak{S}}(\zeta) = \sum_{n=-\infty}^{\infty} (\Psi^* \mathcal{U}^{*n} \Psi) \zeta^{[n]}.$$

We shall refer to the string of coefficients  $\{w_{\mathfrak{S},n}\}_{n \in \mathbb{Z}}$  given by

$$w_{\mathfrak{S},n} := \Psi^* \mathcal{U}^{*n} \Psi \quad (3.4)$$

as the *characteristic moment sequence*.

Let us now introduce the spectral measure  $E_{\mathcal{U}}(\cdot)$  (see e.g. [48]) for  $\mathcal{U}$ ; we then define the *characteristic measure*  $\sigma_{\mathfrak{S}}$  for the unitary scattering system  $\mathfrak{S}$  to be the spectral measure for  $\mathcal{U}$  compressed by the action of  $\Psi$  given by

$$\sigma_{\mathfrak{S}}(\cdot) = \Psi^* E_{\mathcal{U}}(\cdot) \Psi. \quad (3.5)$$

Thus the spectral measure  $E_{\mathcal{U}}$  is a strong Borel measure on the unit circle  $\mathbb{T}$  with values equal to orthogonal projection operators in  $\mathcal{L}(\mathcal{K})$  while the characteristic measure  $\sigma_{\mathfrak{S}}$  is a strong Borel measure on  $\mathbb{T}$  with values equal to positive-semidefinite operators in  $\mathcal{L}(\mathcal{E})$ . Note that the characteristic function  $w_{\mathfrak{S}}(\zeta)$  can be expressed in terms of the characteristic measure  $\sigma_{\mathfrak{S}}$  via the Poisson integral:

$$\begin{aligned} w_{\mathfrak{S}}(\zeta) &= \Psi^* \left[ \int_{\mathbb{T}} ((1 - \bar{\zeta}t)^{-1} + (1 - \zeta\bar{t})^{-1} - 1) E_{\mathcal{U}}(dt) \right] \Psi \\ &= \int_{\mathbb{T}} \mathcal{P}(t, \zeta) \sigma_{\mathfrak{S}}(dt) \end{aligned} \quad (3.6)$$

where

$$\mathcal{P}(t, \zeta) = \frac{1 - |\zeta|^2}{|1 - \bar{\zeta}t|^2} \text{ for } t \in \mathbb{T} \text{ and } \zeta \in \mathbb{D} \quad (3.7)$$

is the classical *Poisson kernel* and where the Lebesgue integral converges in the strong operator topology.

We say that two unitary scattering systems  $(\mathcal{U}, \Psi; \mathcal{K}, \mathcal{E})$  and  $(\mathcal{U}', \Psi'; \mathcal{K}', \mathcal{E})$  (with the same coefficient space  $\mathcal{E}$ ) are *unitarily equivalent* if there is a unitary map  $\tau: \mathcal{K} \rightarrow \mathcal{K}'$  so that

$$\tau \mathcal{U} = \mathcal{U}' \tau, \quad \tau \Psi = \Psi'. \quad (3.8)$$

We say that a unitary scattering system  $\mathfrak{S} = (\mathcal{U}, \Psi; \mathcal{K}, \mathcal{E})$  is *minimal* in case the linear manifold  $\Psi \mathcal{E} \subset \mathcal{K}$  is  $*$ -cyclic for  $\mathcal{U}$ , i.e., the smallest subspace  $\mathcal{K}_0$  containing  $\Psi \mathcal{E}$  and invariant for both  $\mathcal{U}$  and  $\mathcal{U}^*$  is the whole space  $\mathcal{K}$ . The following elementary result makes precise the idea that *the characteristic function is a complete unitary invariant for minimal unitary scattering systems*.

**Proposition 3.1.** *Two unitarily equivalent unitary scattering systems*

$$\mathfrak{S} = (\mathcal{U}, \Psi; \mathcal{K}, \mathcal{E}) \text{ and } \mathfrak{S}' = (\mathcal{U}', \Psi'; \mathcal{K}', \mathcal{E})$$

*have the same characteristic functions ( $w_{\mathfrak{S}}(\zeta) = w_{\mathfrak{S}'}(\zeta)$  for all  $\zeta \in \mathbb{D}$ ).*

*Conversely, if  $\mathfrak{S} = (\mathcal{U}, \Psi; \mathcal{K}, \mathcal{E})$  and  $\mathfrak{S}' = (\mathcal{U}', \Psi'; \mathcal{K}', \mathcal{E})$  are two minimal unitary scattering systems with the same characteristic function, then  $\mathfrak{S}$  and  $\mathfrak{S}'$  are unitarily equivalent.*

*Proof.* This is essentially Theorem 4.1' in [18]. For the reader's convenience, we recall the proof here. If  $\tau: \mathcal{K} \rightarrow \mathcal{K}'$  satisfies the intertwining conditions (3.8), then

$$\begin{aligned} w_{\mathfrak{S}'}(\zeta) &= \Psi'^* [(I - \bar{\zeta}\mathcal{U}')^{-1} + (I - \zeta\mathcal{U}'^*)^{-1} - I] \Psi' \\ &= \Psi'^* \tau^* [(I - \bar{\zeta}\mathcal{U}')^{-1} + (I - \zeta\mathcal{U}'^*)^{-1} - I] \tau \Psi \\ &= \Psi'^* [(I - \bar{\zeta}\mathcal{U})^{-1} + (I - \zeta\mathcal{U}^*)^{-1} - I] \Psi \\ &= w_{\mathfrak{S}}(\zeta). \end{aligned}$$

Conversely, suppose that  $\mathfrak{S}$  and  $\mathfrak{S}'$  are *minimal* unitary scattering systems with the same coefficient space  $\mathcal{E}$  and with identical characteristic functions  $w_{\mathfrak{S}}(\zeta) = w_{\mathfrak{S}'}(\zeta)$  for all  $\zeta \in \mathbb{D}$ . The identity  $w_{\mathfrak{S}}(\zeta) = w_{\mathfrak{S}'}(\zeta)$  between harmonic functions implies that coefficients of powers of  $\zeta$  and of  $\bar{\zeta}$  match up:

$$\Psi^* \mathcal{U}^n \Psi = \Psi'^* \mathcal{U}'^n \Psi' \text{ for } n = 0, \pm 1, \pm 2, \dots$$

Note that

$$\langle \mathcal{U}^n \Psi e, \mathcal{U}^m \Psi \tilde{e} \rangle = \langle \Psi^* \mathcal{U}^{n-m} \Psi e, \tilde{e} \rangle = \langle \Psi'^* \mathcal{U}'^{n-m} \Psi' e, \tilde{e} \rangle = \langle \mathcal{U}'^n \Psi' e, \mathcal{U}'^m \Psi' \tilde{e} \rangle$$

for all  $n, m = 0, \pm 1, \pm 2, \dots$  and  $e, \tilde{e} \in \mathcal{E}$ , and hence the formula

$$\tau: \mathcal{U}^n \Psi e \mapsto \mathcal{U}'^n \Psi' e \quad (3.9)$$

extends by linearity and continuity to define a well-defined isometry from

$$\mathcal{D} = \overline{\text{span}}\{\mathcal{U}^n \Psi e: n \in \mathbb{Z}, e \in \mathcal{E}\}$$

onto

$$\mathcal{R} := \overline{\text{span}}\{\mathcal{U}'^n \Psi' e: n \in \mathbb{Z}, e \in \mathcal{E}\}.$$

Under the assumption that both  $\mathfrak{S}$  and  $\mathfrak{S}'$  are minimal, we see that  $\mathcal{D} = \mathcal{K}$  and  $\mathcal{R} = \mathcal{K}'$ , and hence  $\tau$  is unitary from  $\mathcal{K}$  onto  $\mathcal{K}'$ . From the formula (3.9) for  $\tau$  specialized to the case  $n = 0$ , we see that  $\tau \Psi = \Psi'$ . From the general case of the formula we see that  $\tau \mathcal{U} k = \mathcal{U}' \tau k$  in case  $k \in \mathcal{K}$  has the form  $k = \mathcal{U}^n \Psi^* e$  for some  $n \in \mathbb{Z}$  and  $e \in \mathcal{E}$ . By the minimality assumption on  $\mathfrak{S}$  the span of such elements is dense in  $\mathcal{K}$ , and hence the validity of the intertwining  $\tau \mathcal{U} k = \mathcal{U}' \tau k$  extends to the case of a general element  $k$  of  $\mathcal{K}$ . This concludes the proof of Proposition 3.1.  $\square$

**3.2. The Hellinger space  $\mathcal{L}^\sigma$ .** We mostly follow here definitions and notations of [18]. Earlier expositions were given in [34], [35], [36], [37], [46]. Classical references for the Hellinger integral are [27], [68], [69], [70]; we refer to [63] for an application to stochastic differential equations.

Let  $\mathcal{E}$  be a Hilbert space and let  $\sigma$  be a positive  $\mathcal{L}(\mathcal{E})$ -valued Borel measure on  $\mathbb{T}$ . We define the space  $\mathcal{L}^\sigma$  to be the space of all  $\mathcal{E}$ -valued vector measures  $\nu$  for which there exists a scalar measure  $\mu$  on  $\mathbb{T}$  such that the operator

$$\begin{bmatrix} \mu(\Delta) & \nu(\Delta)^* \\ \nu(\Delta) & \sigma(\Delta) \end{bmatrix} \in \mathcal{L}(\mathbb{C} \oplus \mathcal{E}) \quad (3.10)$$

is positive semidefinite for all Borel subsets  $\Delta \subseteq \mathbb{T}$ . It follows that, for a given  $\nu \in \mathcal{L}^\sigma$  and  $\Delta$  any Borel subset of  $\mathbb{T}$ ,  $\nu(\Delta) \in \text{im } \sigma(\Delta)^{1/2}$  and that the smallest constant  $C_\nu(\Delta)$  which can be substituted for  $\mu(\Delta)$  in (3.10) is

$$C_\nu(\Delta) = \|\sigma(\Delta)^{[-1/2]} \nu(\Delta)\|^2 \quad (3.11)$$

where  $\sigma(\Delta)^{[-1/2]}$  is the Moore-Penrose inverse of  $\sigma(\Delta)^{1/2}$ ; here in general we use the notation  $X^{[-1]}$  for the *Moore-Penrose inverse* of the operator  $X$ :

$$\text{domain } X^{[-1]} = \text{im } X \oplus (\text{im } X)^\perp \text{ with}$$

$$X^{[-1]}x = 0 \text{ if } x \in (\text{im } X)^\perp \text{ and } X^{[-1]}(Xy) = P_{(\ker X)^\perp}y.$$

It can be further shown (see [18] for detail) that there is a unique such dominating scalar measure  $\mu_\nu$  which is *minimal* in the sense that, for any other scalar measure  $\mu$  for which (3.10) holds, then necessarily

$$\mu(\Delta) \geq \mu_\nu(\Delta) \text{ for all } \Delta \subseteq \mathbb{T}$$

This unique measure  $\mu_\nu$  is called the *Hellinger (scalar-valued) measure of the vector measure  $\nu$  with respect to  $\sigma$* . The function  $\Delta \mapsto C_\nu(\Delta)$  in general is not even an additive function of the set  $\Delta$  but merely subadditive, i.e., for  $\Delta_1, \Delta_2$  disjoint Borel subsets of  $\mathbb{T}$ , we are guaranteed only

$$C_\nu(\Delta_1 \cup \Delta_2) \leq C_\nu(\Delta_1) + C_\nu(\Delta_2).$$

Therefore, the Hellinger measure  $\mu_\nu$  is defined as the minimal measure dominating the subadditive function  $C_\nu$ , namely

$$\begin{aligned} \mu_\nu(\Delta) &= \sup \left\{ \sum_{k=1}^n \|\sigma(\Delta_k)^{[-1/2]} \nu(\Delta_k)\|_{\mathcal{E}}^2 : \Delta_1, \dots, \Delta_n \right. \\ &\quad \left. \text{form a finite disjunctiv partition of } \Delta \right\} \\ &= \lim_{\Delta = \dot{\cup}_{k=1}^n \Delta_k} \sum_{k=1}^n \|\sigma(\Delta_k)^{[-1/2]} \nu(\Delta_k)\|_{\mathcal{E}}^2 \end{aligned} \quad (3.12)$$

where the limit is taken along the directed set of finite partitions of  $\Delta$  ordered by refinement. In particular, part of the assertion here is that  $\nu(\Delta) \in \text{im } \sigma(\Delta)^{1/2}$  for each Borel subset  $\Delta$  of  $\mathbb{T}$ .

We next define a norm  $\|\cdot\|_{\mathcal{L}^\sigma}$  by

$$\|\nu\|_{\mathcal{L}^\sigma} = \mu_\nu(\mathbb{T})^{1/2}.$$

Then one can show that  $\mathcal{L}^\sigma$  is a Hilbert space in this norm (see [18]). By polarization of the formula (3.12) we see that the inner product can be expressed as

$$\langle \nu_1, \nu_2 \rangle_{\mathcal{L}^\sigma} = \lim_{\mathbb{T} = \dot{\cup}_{k=1}^n \Delta_k} \sum_{k=1}^n \left\langle \sigma(\Delta_k)^{[-1/2]} \nu_1(\Delta_k), \sigma(\Delta_k)^{[-1/2]} \nu_2(\Delta_k) \right\rangle_{\mathcal{E}}. \quad (3.13)$$

We give below two propositions from [18] that will be used in this paper. Both can be verified for simple functions using formulas (3.12) and (3.13) and then for measurable functions by the limit process.

**Proposition 3.2.** *If  $\nu \in \mathcal{L}^\sigma$  with associated Hellinger scalar-valued measure  $\mu_\nu$  and if  $f$  is a measurable scalar function on  $\mathbb{T}$  for which  $\int_{\mathbb{T}} |f(t)|^2 \mu_\nu(dt) < \infty$ , then the measure  $\nu \cdot f$  given by*

$$(\nu \cdot f)(\Delta) = \int_{\Delta} \nu(dt) f(t)$$

*is in  $\mathcal{L}^\sigma$  with associated Hellinger measure  $\mu_{\nu \cdot f}$  equal to the scalar-valued measure  $\mu_\nu \cdot |f|^2$  given by*

$$(\mu_\nu \cdot |f|^2)(\Delta) = \int_{\Delta} \mu_\nu(dt) |f(t)|^2.$$

**Proposition 3.3.** *Let  $f$  be any  $\mathcal{E}$ -valued measurable function on  $\mathbb{T}$  for which*

$$\int_{\mathbb{T}} \langle \sigma(dt) f(t), f(t) \rangle < \infty,$$

*then the vector measure  $\sigma \cdot f$  given by*

$$(\sigma \cdot f)(\Delta) = \int_{\Delta} d\sigma(t) f(t)$$

*is in  $\mathcal{L}^\sigma$  with  $\mu_{\sigma \cdot f} = f^* \cdot \sigma \cdot f$ , where  $f^* \cdot \sigma \cdot f$  is the scalar measure defined by*

$$(f^* \cdot \sigma \cdot f)(\Delta) = \int_{\Delta} \langle \sigma(dt) f(t), f(t) \rangle_{\mathcal{E}}. \quad (3.14)$$

*Moreover, for every  $\nu \in \mathcal{L}^\sigma$  and every such  $f$  it holds that*

$$\langle \nu, \sigma \cdot f \rangle_{\mathcal{L}^\sigma} = \int_{\mathbb{T}} \langle \nu(dt), f(t) \rangle_{\mathcal{E}}. \quad (3.15)$$

We will need the following uniqueness result.

**Lemma 3.4.** *If  $\sigma$  and  $\tilde{\sigma}$  are two positive  $\mathcal{L}(\mathcal{E})$ -valued measures such that  $\mathcal{L}^\sigma = \mathcal{L}^{\tilde{\sigma}}$  with identity of norms, then  $\sigma = \tilde{\sigma}$ .*

*Proof.* Suppose that  $\mathcal{L}^\sigma = \mathcal{L}^{\tilde{\sigma}}$  with

$$\|\nu\|_{\mathcal{L}^\sigma}^2 = \|\nu\|_{\mathcal{L}^{\tilde{\sigma}}}^2 \text{ for all } \nu \in \mathcal{L}^\sigma = \mathcal{L}^{\tilde{\sigma}}.$$

Then this also holds for  $\chi_\Delta \nu$  with  $\Delta$  an arbitrary Borel set. Then this implies that  $\mu_\nu = \tilde{\mu}_\nu$ , where  $\mu_\nu$  and  $\tilde{\mu}_\nu$  are the Hellinger measures of  $\nu$  with respect to  $\sigma$  and  $\tilde{\sigma}$ . Take  $\nu = \sigma e$ , then, in view of (3.12),  $\mu_{\sigma e} = e^* \sigma e$ , therefore,  $\tilde{\mu}_{\sigma e} = \mu_{\sigma e} = e^* \sigma e$ . This implies that

$$\begin{bmatrix} e^* \sigma e & e^* \tilde{\sigma} \\ \sigma e & \tilde{\sigma} \end{bmatrix} \geq 0.$$

Therefore,

$$\begin{bmatrix} e^* \sigma e & e^* \tilde{\sigma} e \\ e^* \tilde{\sigma} e & e^* \tilde{\sigma} e \end{bmatrix} \geq 0.$$

The latter implies that  $e^* \tilde{\sigma} e \geq e^* \sigma e$ . In other words  $\tilde{\sigma} \geq \sigma$ . The reverse inequality is obtained similarly by taking  $\nu = \tilde{\sigma} e$  and using the fact that  $\mu_{\tilde{\sigma} e} = \tilde{\mu}_{\tilde{\sigma} e} = e^* \tilde{\sigma} e$ . The lemma follows.  $\square$

The connection between unitary scattering systems (3.1) and spaces of vector measures  $\mathcal{L}^\sigma$  (see (3.10)) is as follows. A consequence of Proposition 3.2 is that  $\nu \cdot t \in \mathcal{L}^\sigma$  whenever  $\nu \in \mathcal{L}^\sigma$  with  $\|\nu \cdot t\|_{\mathcal{L}^\sigma} = \|\nu\|_{\mathcal{L}^\sigma}$  (here  $t$  stands for the function  $f(t) = t$  on  $\mathbb{T}$ ); since the same story holds for  $\nu \cdot t^{-1} = \nu \cdot \bar{t}$ , we see that the operator

$$\mathcal{U}_\sigma: \nu \mapsto \nu \cdot t$$

is unitary on  $\mathcal{L}^\sigma$  with spectral measure  $E_{\mathcal{U}_\sigma}$  given by

$$E_{\mathcal{U}_\sigma}(\Delta): \nu \mapsto \nu \cdot \chi_\Delta \text{ for } \nu \in \mathcal{L}^\sigma.$$

It follows from the definition of the Hellinger space and the Hellinger measure that  $\sigma e \in \mathcal{L}^\sigma$  and that  $\mu_{\sigma e}(\Delta) = \langle \sigma(\Delta) e, e \rangle$ . In particular,

$$\|\sigma e\|_{\mathcal{L}^\sigma}^2 = \langle \sigma(\mathbb{T}) e, e \rangle_{\mathcal{E}} \leq \|\sigma(\mathbb{T})\| \cdot \|e\|^2.$$

Therefore, the operator  $\Psi_\sigma$  given by

$$\Psi_\sigma: e \mapsto \sigma \cdot e$$

is a bounded linear operator from  $\mathcal{E}$  into  $\mathcal{L}^\sigma$ . Hence the collection

$$\mathfrak{S}_\sigma = (\mathcal{U}_\sigma, \Psi_\sigma; \mathcal{L}^\sigma, \mathcal{E}) \quad (3.16)$$

is a unitary scattering system for any positive  $\mathcal{L}(\mathcal{E})$ -valued measure  $\sigma$ . Moreover, an easy computation

$$\begin{aligned} \langle \Psi_\sigma^* E_{\mathcal{U}_\sigma}(\Delta) \Psi_\sigma e, e \rangle_{\mathcal{L}^\sigma} &= \langle E_{\mathcal{U}_\sigma}(\Delta) \sigma e, \sigma e \rangle_{\mathcal{L}^\sigma} \\ &= \langle \sigma \cdot \chi_\Delta e, \sigma e \rangle_{\mathcal{L}^\sigma} \\ &= \langle \sigma(\Delta) e, e \rangle_{\mathcal{E}} \text{ by formula (3.13)} \end{aligned}$$

shows that we recover the preassigned positive operator measure  $\sigma$  as the characteristic measure  $\Psi_\sigma^* E_{\mathcal{U}_\sigma}(\cdot) \Psi_\sigma$  for the unitary scattering system  $\mathfrak{S}_\sigma$ . In particular, *for any positive  $\mathcal{L}(\mathcal{E})$ -valued Borel measure  $\sigma$ , there exists a unitary scattering system  $\mathfrak{S}_\sigma$  having characteristic measure equal to  $\sigma$* . Since positive operator measures  $\sigma$  are in one-to-one correspondence with positive operator-valued harmonic functions  $w$  via the Poisson representation  $w(\zeta) = \int_{\mathbb{T}} \mathcal{P}(t, \zeta) \sigma(dt)$  (where  $\mathcal{P}$  is the Poisson kernel (3.7)), we also see that *given any positive operator-valued harmonic function  $w$ , there is a unitary scattering system  $\mathfrak{S}_w$  having characteristic function (3.2) equal to  $w$* . The following result (Theorem 4.1 of [18]) is the converse to this statement.

**Theorem 3.5.** *Let  $\mathfrak{S} = (\mathcal{U}, \Psi; \mathcal{K}, \mathcal{E})$  be a unitary scattering system as in (3.1). Let  $\sigma$  be the associated characteristic measure*

$$\sigma(\cdot) = \Psi^* E_{\mathcal{U}}(\cdot) \Psi.$$

*For  $k \in \mathcal{K}$ , define an  $\mathcal{E}$ -valued Borel measure  $\nu_k$  on  $\mathbb{T}$  by*

$$\nu_k = \Psi^* E_{\mathcal{U}} k.$$

*Then the operator  $\Phi^{\wedge m}$  (the vector-measure valued Fourier representation operator for  $\mathfrak{S}$ ) given by*

$$\Phi^{\wedge m}: k \mapsto \nu_k \quad (3.17)$$

*is a coisometry from  $\mathcal{K}$  onto  $\mathcal{L}^\sigma$  satisfying the intertwining relations*

$$\Phi^{\wedge m} \mathcal{U} = \mathcal{U}_\sigma \Phi^{\wedge m}, \quad \Phi^{\wedge m} \Psi = \Psi_\sigma$$

*with initial space equal to the smallest reducing subspace for  $\mathcal{U}$  containing  $\text{im } \Psi$ . In particular, if  $\mathfrak{S}$  is minimal, then the unitary scattering system  $\mathfrak{S}$  is unitarily equivalent (via the unitary operator  $\Phi^{\wedge m}: \mathcal{K} \rightarrow \mathcal{L}^\sigma$ ) to the model unitary scattering system  $\mathfrak{S}_\sigma$  given by (3.16).*

**Remark 3.6.** A space of vector-valued harmonic functions on the unit disk  $\mathbb{D}$  can be associated to the space  $\mathcal{L}^\sigma$  of the vector-valued measures on the circle  $\mathbb{T}$  via integration against the Poisson kernel. The formalism here can be translated from measures on the circle to harmonic functions on the disk. Such an analysis is worked out in [35, 36, 34, 37, 18, 26].

We also mention that a functional model using formal reproducing kernel Hilbert spaces (rather than measures and Hellinger spaces) for analogues of unitary scattering systems as defined in Subsection 3.1, where the unitary operator  $\mathcal{U}$  is replaced by (1) a tuple of commuting unitary operators  $(\mathcal{U}_1, \dots, \mathcal{U}_d)$ , or (2) a row-unitary operator  $[\mathcal{U}_1 \ \cdots \ \mathcal{U}_d]$  (i.e., a representation of the Cuntz algebra), is presented in [16].

In the sequel we shall have need of the following result concerning orthogonal decompositions of Hellinger spaces  $\mathcal{L}^\sigma$  (see Theorem 2.8 in [18]).



**Theorem 3.7.** *Suppose that the coefficient Hilbert space  $\mathcal{E}$  has an orthogonal direct-sum decomposition  $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$  and that  $\sigma$  is an  $\mathcal{L}(\mathcal{E})$ -valued positive measure with block-matrix decomposition*

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \text{ where } \sigma_{ij}(\Delta) \in \mathcal{L}(\mathcal{E}_j, \mathcal{E}_i) \text{ for each Borel set } \Delta \text{ and for } i, j = 1, 2.$$

Define subspaces  $\mathcal{L}_1^\sigma$  and  $\mathcal{L}_2^\sigma$  of  $\mathcal{L}^\sigma$  by

$$\begin{aligned} \mathcal{L}_1^\sigma &= \left\{ \nu = \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \in \mathcal{L}^\sigma : \|\nu\|_{\mathcal{L}^\sigma} = \|\nu_1\|_{\mathcal{L}^{\sigma_{11}}} \right\}, \\ \mathcal{L}_2^\sigma &= \left\{ \nu = \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \in \mathcal{L}^\sigma : \nu_1 = 0 \right\}. \end{aligned}$$

Then:

- (1)  $\mathcal{L}_1^\sigma = \mathcal{L}^\sigma - \text{clos.} \left\{ \sigma \begin{bmatrix} I \\ 0 \end{bmatrix} p : p \in \mathcal{E}_1[t, t^{-1}] \right\}$  where  $\mathcal{E}_1[t, t^{-1}]$  is the space of trigonometric polynomials with coefficients in  $\mathcal{E}_1$ , and
- (2)  $\mathcal{L}_2^\sigma = \mathcal{L}^\sigma \ominus \mathcal{L}_1^\sigma = \left\{ \begin{bmatrix} 0 \\ q \end{bmatrix} : q \in \mathcal{L}^{\sigma_{11}^\perp} \right\}$  where  $\sigma_{11}^\perp = \sigma_{22} - \sigma_{21}\sigma_{11}^{[-1]}\sigma_{12}$  is the measure Schur-complement of  $\sigma_{11}$  inside  $\sigma$  in the sense of [18, Section 2]. In particular,

$$\mathcal{L}_1^\sigma = \mathcal{L}^\sigma \text{ if and only if } \sigma_{11}^\perp = 0. \quad (3.18)$$

*Proof.* We prove here Statement (1) which is contained implicitly in Theorem 2.8 of [18].

For  $p \in \mathcal{E}_1[t, t^{-1}]$ , the computation

$$\begin{aligned} \left\langle \sigma \begin{bmatrix} I \\ 0 \end{bmatrix} p, \sigma \begin{bmatrix} I \\ 0 \end{bmatrix} p \right\rangle_{\mathcal{L}^\sigma} &= \int_{\mathbb{T}} p(t)^* \begin{bmatrix} I & 0 \end{bmatrix} \sigma(dt) \begin{bmatrix} I \\ 0 \end{bmatrix} p(t) \\ &= \int_{\mathbb{T}} p(t)^* \sigma_{11}(dt) p_{11}(t) \\ &= \langle \sigma_{11} p, \sigma_{11} p \rangle_{\mathcal{L}^{\sigma_{11}}} \end{aligned}$$

shows that the map

$$\sigma_{11} p \mapsto \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \end{bmatrix} p \text{ for } p \in \mathcal{E}_1[t, t^{-1}]$$

embeds a dense subset of  $\mathcal{L}^{\sigma_{11}}$  isometrically into  $\mathcal{L}_1^\sigma$ . The fact that the image of this map is dense in  $\mathcal{L}_1^\sigma$  follows from the definition of  $\mathcal{L}_1^\sigma$  and from the fact that  $\{\sigma_{11} p : p \in \mathcal{E}_1[t, t^{-1}]\}$  is dense in  $\mathcal{L}^{\sigma_{11}}$ . Statement (2) is proved in Theorem 2.8 of [18].  $\square$

#### 4. INTERTWINERS AND UNITARY COUPLINGS OF UNITARY OPERATORS

In this section we present some preliminary material on unitary couplings due originally to Adamjan and Arov [1] which is needed for a reformulation of the Lifting Problem to be presented in the next section.

Suppose that we are given unitary operators  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$ . We say that the collection  $(\mathcal{U}, i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K})$  is an *Adamjan-Arov unitary coupling* (or, more briefly, *AA-unitary coupling*) of  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$  if  $\mathcal{U}$  is a unitary operator on

the Hilbert space  $\mathcal{K}$  and  $i_{\mathcal{K}'}: \mathcal{K}' \rightarrow \mathcal{K}$  and  $i_{\mathcal{K}''}: \mathcal{K}'' \rightarrow \mathcal{K}$  are isometric embeddings of  $\mathcal{K}'$  and  $\mathcal{K}''$  respectively into  $\mathcal{K}$  such that

$$i_{\mathcal{K}'}\mathcal{U}' = \mathcal{U}i_{\mathcal{K}'}, \quad i_{\mathcal{K}''}\mathcal{U}'' = \mathcal{U}i_{\mathcal{K}''}. \quad (4.1)$$

In this case it is clear that  $Y = i_{\mathcal{K}''}^* i_{\mathcal{K}'}: \mathcal{K}' \rightarrow \mathcal{K}''$  is contractive ( $\|Y\| \leq 1$ ) and that  $Y$  intertwines  $\mathcal{U}'$  with  $\mathcal{U}''$  since

$$Y\mathcal{U}' = i_{\mathcal{K}''}^* i_{\mathcal{K}'}\mathcal{U}' = i_{\mathcal{K}''}^* \mathcal{U}i_{\mathcal{K}'} = \mathcal{U}'' i_{\mathcal{K}''}^* i_{\mathcal{K}'} = \mathcal{U}'' Y.$$

The following theorem provides a converse to this observation. To formalize the ideas, let us say that the AA-unitary coupling  $(\mathcal{U}, i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K})$  is *minimal* in case

$$\text{im } i_{\mathcal{K}'} + \text{im } i_{\mathcal{K}''} \text{ is dense in } \mathcal{K} \quad (4.2)$$

and that two AA-unitary couplings  $(\mathcal{U}, i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K})$  and  $(\tilde{\mathcal{U}}, \tilde{i}_{\mathcal{K}'}, \tilde{i}_{\mathcal{K}''}; \tilde{\mathcal{K}})$  of  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$  are *unitarily equivalent* if there is a unitary operator  $\tau: \mathcal{K} \rightarrow \tilde{\mathcal{K}}$  such that

$$\tau\mathcal{U} = \tilde{\mathcal{U}}\tau, \quad \tau i_{\mathcal{K}'} = \tilde{i}_{\mathcal{K}'}, \quad \tau i_{\mathcal{K}''} = \tilde{i}_{\mathcal{K}''}. \quad (4.3)$$

Then we have the following fundamental connection between AA-unitary couplings and contractive intertwiners of two given unitary operators (see e.g. [1] and [19]).

**Theorem 4.1.** *Suppose that we are given two unitary operators  $\mathcal{U}'$  and  $\mathcal{U}''$  on Hilbert spaces  $\mathcal{K}'$  and  $\mathcal{K}''$ , respectively. Then there is a one-to-one correspondence between unitary equivalence classes of minimal AA-unitary couplings  $(\mathcal{U}, i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K})$  and contractive intertwiners  $Y: \mathcal{K}' \rightarrow \mathcal{K}''$  of  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$ . More precisely:*

- (1) *Suppose that  $\mathfrak{A} := (\mathcal{U}, i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K})$  is an AA-unitary coupling of  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$ . Define an operator  $Y = Y(\mathfrak{A}): \mathcal{K}' \rightarrow \mathcal{K}''$  via*

$$Y = Y(\mathfrak{A}) = i_{\mathcal{K}''}^* i_{\mathcal{K}'}. \quad (4.4)$$

*Then  $Y$  is a contractive intertwiner of  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$ . Equivalent AA-unitary couplings produce the same intertwiner  $Y$  via (4.4).*

- (2) *Suppose that  $Y: \mathcal{K}' \rightarrow \mathcal{K}''$  is a contractive intertwiner of  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$ . Define a Hilbert space  $\mathcal{K} := \mathcal{K}'' *_Y \mathcal{K}'$  as the completion of the space  $\begin{bmatrix} \mathcal{K}'' \\ \mathcal{K}' \end{bmatrix}$  in the inner product*

$$\left\langle \begin{bmatrix} k'' \\ k' \end{bmatrix}, \begin{bmatrix} h'' \\ h' \end{bmatrix} \right\rangle_{\mathcal{K}'' *_Y \mathcal{K}'} = \left\langle \begin{bmatrix} I_{\mathcal{K}''} & Y \\ Y^* & I_{\mathcal{K}'} \end{bmatrix} \begin{bmatrix} k'' \\ k' \end{bmatrix}, \begin{bmatrix} h'' \\ h' \end{bmatrix} \right\rangle_{\mathcal{K}'' \oplus \mathcal{K}'} \quad (4.5)$$

*for  $k'', h'' \in \mathcal{K}''$  and  $k', h' \in \mathcal{K}'$  (with pairs  $\begin{bmatrix} k'' \\ k' \end{bmatrix}$  with zero self inner product identified with 0). Define an operator  $\mathcal{U} = \mathcal{U}'' *_Y \mathcal{U}'$  densely on  $\mathcal{K}$  by*

$$\mathcal{U}: \begin{bmatrix} k'' \\ k' \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{U}'' k'' \\ \mathcal{U}' k' \end{bmatrix} \quad (4.6)$$

*together with inclusion maps  $i_{\mathcal{K}'}: \mathcal{K}' \rightarrow \mathcal{K}$  and  $i_{\mathcal{K}''}: \mathcal{K}'' \rightarrow \mathcal{K}$  given by*

$$i_{\mathcal{K}'}: k' \mapsto \begin{bmatrix} 0 \\ k' \end{bmatrix}, \quad i_{\mathcal{K}''}: k'' \mapsto \begin{bmatrix} k'' \\ 0 \end{bmatrix}. \quad (4.7)$$

*Then the resulting collection*

$$\mathfrak{A} = \mathfrak{A}(Y) := (\mathcal{U}'' *_Y \mathcal{U}', i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K}'' *_Y \mathcal{K}') \quad (4.8)$$

*is a minimal AA-unitary coupling of  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$  such that we recover  $Y$  as  $Y = i_{\mathcal{K}''}^* i_{\mathcal{K}'}$ . Any minimal AA-unitary coupling  $(\mathcal{U}, i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K})$*

of  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$  with intertwiner  $Y$  as in (4.4) is unitarily equivalent to the one defined by (4.5), (4.6), (4.7).

Moreover, the maps  $\mathfrak{A} \mapsto Y(\mathfrak{A})$  in (1) and  $Y \mapsto \mathfrak{A}(Y)$  in (2) are inverse to each other and set up a one-to-one correspondence between contractive intertwiners  $Y$  and unitary equivalence classes of minimal AA-unitary couplings of  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$ .

*Proof.* The proof of part (1) was already observed in the discussion preceding the statement of the theorem.

Conversely, suppose that  $Y: \mathcal{K}' \rightarrow \mathcal{K}''$  is any contractive intertwiner of  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$  and let  $(\mathcal{U}' *_Y \mathcal{U}'', i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K}' *_Y \mathcal{K}'')$  be the AA-unitary coupling of  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$  given by (4.8). From the form of the inner product (4.5), we see that the maps  $i_{\mathcal{K}'}: \mathcal{K}' \rightarrow \mathcal{K}$  and  $i_{\mathcal{K}''}: \mathcal{K}'' \rightarrow \mathcal{K}$  given by (4.7) are isometric. By the definition of  $\mathcal{K}$  as the completion of  $\begin{bmatrix} \mathcal{K}'' \\ \mathcal{K}' \end{bmatrix}$  in the  $\mathcal{K}'' *_Y \mathcal{K}'$ -inner product, we see that the span of the images  $\text{im } i_{\mathcal{K}'} + \text{im } i_{\mathcal{K}''}$  is dense in  $\mathcal{K} := \mathcal{K}' *_Y \mathcal{K}''$  by construction. By using the intertwining condition  $Y\mathcal{U}' = \mathcal{U}''Y$  together with the unitary property of  $\mathcal{U}''$  and  $\mathcal{U}'$ , we see that

$$\begin{aligned} & \left\langle \begin{bmatrix} I_{\mathcal{K}''} & Y \\ Y^* & I_{\mathcal{K}'} \end{bmatrix} \begin{bmatrix} \mathcal{U}'' k'' \\ \mathcal{U}' k' \end{bmatrix}, \begin{bmatrix} \mathcal{U}'' h'' \\ \mathcal{U}' h' \end{bmatrix} \right\rangle_{\mathcal{K}'' \oplus \mathcal{K}'} \\ &= \left\langle \begin{bmatrix} \mathcal{U}'' & 0 \\ 0 & \mathcal{U}' \end{bmatrix} \begin{bmatrix} I_{\mathcal{K}''} & Y \\ Y^* & I_{\mathcal{K}'} \end{bmatrix} \begin{bmatrix} k'' \\ k' \end{bmatrix}, \begin{bmatrix} \mathcal{U}'' & 0 \\ 0 & \mathcal{U}' \end{bmatrix} \begin{bmatrix} h'' \\ h' \end{bmatrix} \right\rangle_{\mathcal{K}'' \oplus \mathcal{K}'} \\ &= \left\langle \begin{bmatrix} I_{\mathcal{K}''} & Y \\ Y^* & I_{\mathcal{K}'} \end{bmatrix} \begin{bmatrix} k'' \\ k' \end{bmatrix}, \begin{bmatrix} h'' \\ h' \end{bmatrix} \right\rangle_{\mathcal{K}'' \oplus \mathcal{K}'} \end{aligned}$$

and hence the operator

$$\mathcal{U}: \begin{bmatrix} k'' \\ k' \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{U}'' k'' \\ \mathcal{U}' k' \end{bmatrix} \quad (4.9)$$

extends to define a unitary operator on  $\mathcal{K} = \mathcal{K}'' *_Y \mathcal{K}'$ . We have thus verified that  $(\mathcal{U}' *_Y \mathcal{U}'', i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K}' *_Y \mathcal{K}'')$  defined as in (4.8) is a minimal AA-unitary coupling of  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$ , and statement (2) of the theorem follows. From the form of the inner product, we see that we recover  $Y$  as  $Y = i_{\mathcal{K}''}^* i_{\mathcal{K}'}$ .

Suppose now that  $(\mathcal{U}, i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K})$  is any AA-unitary coupling of  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$  and we set  $Y = i_{\mathcal{K}''}^* i_{\mathcal{K}'}: \mathcal{K}' \rightarrow \mathcal{K}''$ . For  $k', \ell' \in \mathcal{K}'$  and  $k'', \ell'' \in \mathcal{K}''$ , we compute

$$\begin{aligned} & \left\langle \begin{bmatrix} I & Y \\ Y^* & I \end{bmatrix} \begin{bmatrix} k'' \\ k' \end{bmatrix}, \begin{bmatrix} \ell'' \\ \ell' \end{bmatrix} \right\rangle = \langle k'' + i_{\mathcal{K}''}^* i_{\mathcal{K}'} k', \ell'' \rangle_{\mathcal{K}''} + \langle i_{\mathcal{K}'}^* i_{\mathcal{K}''} k'' + k', \ell' \rangle_{\mathcal{K}'} \\ &= \langle k'', \ell'' \rangle_{\mathcal{K}''} + \langle i_{\mathcal{K}'} k', i_{\mathcal{K}''} \ell'' \rangle_{\mathcal{K}} + \langle i_{\mathcal{K}''} k'', i_{\mathcal{K}'} \ell' \rangle_{\mathcal{K}} + \langle k', \ell' \rangle_{\mathcal{K}'} \\ &= \langle i_{\mathcal{K}''} k'' + i_{\mathcal{K}'} k', i_{\mathcal{K}''} \ell'' + i_{\mathcal{K}'} \ell' \rangle_{\mathcal{K}}. \end{aligned}$$

We conclude that the map

$$\tau: \begin{bmatrix} k'' \\ k' \end{bmatrix} \mapsto i_{\mathcal{K}''} k'' + i_{\mathcal{K}'} k'$$

maps the dense subspace  $\begin{bmatrix} \mathcal{K}'' \\ \mathcal{K}' \end{bmatrix}$  of  $\mathcal{K}'' *_Y \mathcal{K}'$  onto  $\text{im } i_{\mathcal{K}'} + \text{im } i_{\mathcal{K}''}$ . Hence  $\tau$  extends to an isometric mapping of all of  $\mathcal{K}'' *_Y \mathcal{K}'$  onto the closure of  $\text{im } i_{\mathcal{K}''} + \text{im } i_{\mathcal{K}'}$  and hence  $\tau$  is unitary exactly when  $(\mathcal{U}, i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K})$  is a minimal AA-unitary coupling. Moreover, from the form of  $\tau$  it is easily verified that  $\tau(\mathcal{U}' *_Y \mathcal{U}'') = \mathcal{U}\tau$ . By definition of  $\tau$ , it transforms the embeddings of  $\mathcal{K}'$  and  $\mathcal{K}''$  into  $\mathcal{K}'' *_Y \mathcal{K}'$  to the

embeddings of  $\mathcal{K}'$  and  $\mathcal{K}''$  into  $\mathcal{K}$ . In this way we see that the above correspondence between contractive intertwiners and minimal AA-unitary couplings is bijective.  $\square$

Given an AA-unitary coupling  $(\mathcal{U}, i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K})$  of the unitary operators  $(\mathcal{U}'', \mathcal{K}'')$  and  $(\mathcal{U}', \mathcal{K}')$  and two subspaces  $\mathcal{G}' \subset \mathcal{K}'$  and  $\mathcal{G}'' \subset \mathcal{K}''$  which are  $*$ -cyclic for  $\mathcal{U}'$  and  $\mathcal{U}''$  respectively, let  $i_{\mathcal{G}'}: \mathcal{G}' \rightarrow \mathcal{K}$  and  $i_{\mathcal{G}''}: \mathcal{G}'' \rightarrow \mathcal{K}$  be the compositions of the inclusion of  $\mathcal{G}'$  into  $\mathcal{K}'$  with the inclusion of  $\mathcal{K}'$  into  $\mathcal{K}$  and of the inclusion of  $\mathcal{G}''$  into  $\mathcal{K}''$  with the inclusion of  $\mathcal{K}''$  into  $\mathcal{K}$ , respectively:

$$\begin{aligned} i_{\mathcal{G}'} &:= i_{\mathcal{G}' \rightarrow \mathcal{K}} = i_{\mathcal{K}' \rightarrow \mathcal{K}} i_{\mathcal{G}' \rightarrow \mathcal{K}'}, \\ i_{\mathcal{G}''} &:= i_{\mathcal{G}'' \rightarrow \mathcal{K}} = i_{\mathcal{K}'' \rightarrow \mathcal{K}} i_{\mathcal{G}'' \rightarrow \mathcal{K}''}. \end{aligned}$$

Then we may view

$$\mathfrak{S}_{AA} := (\mathcal{U}, [i_{\mathcal{G}''} \ i_{\mathcal{G}'}]; \mathcal{K}, \mathcal{G}'' \oplus \mathcal{G}')$$

as a unitary scattering system with characteristic measure

$$\sigma_{\mathfrak{S}_{AA}}(\cdot) = \begin{bmatrix} i_{\mathcal{G}''}^* \\ i_{\mathcal{G}'}^* \end{bmatrix} E_{\mathcal{U}}(\cdot) \begin{bmatrix} i_{\mathcal{G}''} & i_{\mathcal{G}'} \end{bmatrix}, \quad (4.10)$$

characteristic function

$$\widehat{w}_{\mathfrak{S}_{AA}}(\zeta) = \begin{bmatrix} i_{\mathcal{G}''}^* \\ i_{\mathcal{G}'}^* \end{bmatrix} [(I - \bar{\zeta}\mathcal{U})^{-1} + (I - \zeta\mathcal{U}^*)^{-1} - I] \begin{bmatrix} i_{\mathcal{G}''} & i_{\mathcal{G}'} \end{bmatrix}$$

and characteristic moment-sequence

$$\{w_{\mathfrak{S}_{AA}}(n)\}_{n \in \mathbb{Z}} \text{ with } w_{\mathfrak{S}_{AA}}(n) = \begin{bmatrix} i_{\mathcal{G}''}^* \\ i_{\mathcal{G}'}^* \end{bmatrix} \mathcal{U}^{*n} \begin{bmatrix} i_{\mathcal{G}''} & i_{\mathcal{G}'} \end{bmatrix}.$$

We note that the  $(1,2)$ -entry in the  $n$ -th characteristic moment  $w_{\mathfrak{S}_{AA},n}$  is closely associated with the intertwiner  $Y = i_{\mathcal{K}''}^* i_{\mathcal{K}}$  associated with the AA-unitary coupling  $(\mathcal{U}, i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K})$ :

$$\langle [w_{\mathfrak{S}_{AA}}(n)]_{12} g', g'' \rangle = \langle i_{\mathcal{G}''}^* \mathcal{U}^{*n} i_{\mathcal{G}'} g', g'' \rangle = \langle Y \mathcal{U}'^{*n} g', g'' \rangle = \langle Y g', \mathcal{U}''^{*n} g'' \rangle.$$

Given any contractive intertwiner  $Y: \mathcal{K}' \rightarrow \mathcal{K}''$  of  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$ , we refer to the bilateral sequence of operators  $\{w_Y(n)\}_{n \in \mathbb{Z}}$  given by

$$w_Y(n) = [w_{\mathfrak{S}_{AA}}(n)]_{12} = i_{\mathcal{G}'' \rightarrow \mathcal{K}''}^* \mathcal{U}''^{*n} Y i_{\mathcal{G}' \rightarrow \mathcal{K}'} = i_{\mathcal{G}'' \rightarrow \mathcal{K}''}^* Y \mathcal{U}'^{*n} i_{\mathcal{G}' \rightarrow \mathcal{K}'} \quad (4.11)$$

as the *symbol* (associated with given subspaces  $\mathcal{G}' \subset \mathcal{K}'$  and  $\mathcal{G}'' \subset \mathcal{K}''$ ) of the intertwiner  $Y$ . If  $\mathcal{G}'$  is  $*$ -cyclic for  $\mathcal{U}'$  and  $\mathcal{G}''$  is  $*$ -cyclic for  $\mathcal{U}''$ , then the subspaces

$$\mathcal{K}'_0 = \overline{\text{span}}\{\mathcal{U}'^n g': g' \in \mathcal{G}' \text{ and } n \in \mathbb{Z}\}, \quad \mathcal{K}''_0 = \overline{\text{span}}\{\mathcal{U}''^m g'': g'' \in \mathcal{G}'' \text{ and } m \in \mathbb{Z}\}$$

are equal to all of  $\mathcal{K}'$  and  $\mathcal{K}''$  respectively and the observation

$$\langle Y \mathcal{U}'^n g', \mathcal{U}''^m g'' \rangle = \langle w_Y(m-n) g', g'' \rangle_{\mathcal{G}''}$$

shows that there is a one-to-one correspondence between symbols  $w_Y$  (with respect to the two  $*$ -cyclic subspaces  $\mathcal{G}'$  and  $\mathcal{G}''$ ) of  $Y$  and the associated contractive intertwiners  $Y$ . Moreover, we have the following characterization of which bilateral  $\mathcal{L}(\mathcal{G}', \mathcal{G}'')$ -valued sequences  $w = \{w(n)\}_{n \in \mathbb{Z}}$  arise as the symbol  $w = w_Y$  for some contractive intertwiner  $Y$ .

**Theorem 4.2.** Suppose that  $\{w(n)\}$  is the symbol (4.11) for a contractive intertwiner  $Y$  of the unitary operators  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$  associated with  $*$ -cyclic subspaces  $\mathcal{G}' \subset \mathcal{K}'$  and  $\mathcal{G}'' \subset \mathcal{K}''$ . Then  $\{w(n)\}_{n \in \mathbb{Z}}$  is the sequence of trigonometric moments

$$w(n) = \int_{\mathbb{T}} t^{-n} \widehat{w}(dt)$$

associated with an  $\mathcal{L}(\mathcal{G}', \mathcal{G}'')$ -valued measure  $\widehat{w}$  (equal to the Fourier transform of  $\{w(n)\}_{n \in \mathbb{Z}}$ ) such that

$$\sigma := \begin{bmatrix} \sigma'' & \widehat{w} \\ \widehat{w}^* & \sigma' \end{bmatrix} \text{ is a positive } \mathcal{L}(\mathcal{G}', \mathcal{G}'') \text{ - valued measure} \quad (4.12)$$

where we have set

$$\sigma' = i_{\mathcal{G}'}^* E_{\mathcal{U}'}(\cdot) i_{\mathcal{G}'}, \quad \sigma'' = i_{\mathcal{G}''}^* E_{\mathcal{U}''}(\cdot) i_{\mathcal{G}''}. \quad (4.13)$$

Conversely, the inverse Fourier transform

$$w(n) := \int_{\mathbb{T}} t^{-n} \widehat{w}(dt)$$

of any  $\mathcal{L}(\mathcal{G}', \mathcal{G}'')$ -valued measure  $\widehat{w}$  on  $\mathbb{T}$  which in addition satisfies (4.12) is the symbol (associated with the subspaces  $\mathcal{G}'$  and  $\mathcal{G}''$ ) for a uniquely determined contractive intertwiner  $Y: \mathcal{K}' \rightarrow \mathcal{K}''$ .

*Proof.* The forward direction of the theorem is an immediate consequence of the results preceding the theorem.

For the converse we use the Hellinger model. Suppose that  $\widehat{w}$  is a vector measure such that (4.12) holds, where  $\sigma'$  and  $\sigma''$  are the positive measures given by (4.13). Consider the Hellinger space  $\mathcal{L}^\sigma$  (see Section 3.2) with operator  $\mathcal{U}^\sigma$  being multiplication by the independent variable  $t$ . We define the embeddings

$$i_{\mathcal{K}'}: \mathcal{K}' \rightarrow \mathcal{L}^\sigma \text{ and } i_{\mathcal{K}''}: \mathcal{K}'' \rightarrow \mathcal{L}^\sigma$$

as follows: first we map  $\mathcal{K}'$  onto  $\mathcal{L}^{\sigma'}$  and  $\mathcal{K}''$  onto  $\mathcal{L}^{\sigma''}$  by Fourier representations

$$k' \mapsto i_{\mathcal{G}'}^* E'(dt) k', \quad k'' \mapsto i_{\mathcal{G}''}^* E''(dt) k''.$$

Then we embed  $\mathcal{L}^{\sigma'}$  and  $\mathcal{L}^{\sigma''}$  into  $\mathcal{L}^\sigma$  by

$$\sigma' p' \mapsto \begin{bmatrix} \widehat{w} \\ \sigma' \end{bmatrix} p', \quad \sigma'' p'' \mapsto \begin{bmatrix} \sigma'' \\ \widehat{w}^* \end{bmatrix} p''$$

for arbitrary vector trigonometric polynomials  $p', p''$ . As it was shown in [18]

$$\text{im } i_{\mathcal{K}'} + \text{im } i_{\mathcal{K}''} \text{ is dense in } \mathcal{L}^\sigma.$$

Thus we get a minimal AA-unitary coupling of  $\mathcal{U}'$  and  $\mathcal{U}''$ . The symbol of the contractive intertwiner associated with this AA-unitary coupling is just the trigonometric-moment sequence of the originally given measure  $\widehat{w}$ . Since the definition of symbol (4.11) can be rephrased as

$$\langle w_Y(n-m) g', g'' \rangle = \langle Y \mathcal{U}'^{*n} g', \mathcal{U}''^{*m} g'' \rangle, \quad (4.14)$$

and the sets  $\{\mathcal{U}'^{*n} g'\}$  and  $\{\mathcal{U}''^{*m} g''\}$  ( $n$  running over  $\mathbb{Z}$ ,  $g'$  over  $\mathcal{G}'$  and  $g''$  over  $\mathcal{G}''$ ) have dense span in  $\mathcal{K}'$  and  $\mathcal{K}''$ , respectively, we see that the correspondence between intertwiners  $Y$  and symbols  $\{w(n)\}_{n \in \mathbb{Z}}$  is one-to-one. Moreover, the correspondence between symbols  $\{w(n)\}_{n \in \mathbb{Z}}$  and their Fourier transforms  $\widehat{w}$  is also one-to-one.  $\square$

**Remark 4.3.** The proof of Theorem 4.2 in fact shows that, given an AA-unitary coupling  $(\mathcal{U}, i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K})$  of two unitary operators  $(\mathcal{U}'', \mathcal{K}'')$  and  $(\mathcal{U}', \mathcal{K}')$  together with a choice of  $*$ -cyclic subspaces  $\mathcal{G}' \subset \mathcal{K}'$  and  $\mathcal{G}'' \subset \mathcal{K}''$ , then the AA-unitary coupling  $(\mathcal{U}, i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K})$  is unitarily equivalent to the Hellinger-model AA-unitary coupling

$$(\mathcal{U}^\sigma, i_{\mathcal{K}' \rightarrow \mathcal{L}^\sigma}, i_{\mathcal{K}'' \rightarrow \mathcal{L}^\sigma}; \mathcal{L}^\sigma)$$

where  $\sigma = \sigma_{\mathfrak{S}_{AA}}$  is given by (4.10).

## 5. LIFTINGS AND UNITARY EXTENSIONS OF AN ISOMETRY DEFINED BY THE PROBLEM DATA

In this section we discuss, following [2, 19, 4, 5, 34, 37, 35, 36, 43, 44, 45], how solutions of the lifting problem can be identified with unitary extensions of a certain (partially defined) isometry  $V$  which is constructed directly from the problem data. Introduce a Hilbert space  $\mathcal{H}_0$  by

$$\mathcal{H}_0 = \text{clos} \begin{bmatrix} \mathcal{K}''_- \\ \mathcal{K}'_+ \end{bmatrix} \quad (5.1)$$

with inner product given by

$$\left\langle \begin{bmatrix} k''_- \\ k'_+ \end{bmatrix}, \begin{bmatrix} \ell''_- \\ \ell'_+ \end{bmatrix} \right\rangle_{\mathcal{H}_0} = \left\langle \begin{bmatrix} I & X \\ X^* & I \end{bmatrix} \begin{bmatrix} k''_- \\ k'_+ \end{bmatrix}, \begin{bmatrix} \ell''_- \\ \ell'_+ \end{bmatrix} \right\rangle_{\mathcal{K}''_- \oplus \mathcal{K}'_+}. \quad (5.2)$$

Special subspaces of  $\mathcal{H}_0$  are of interest:

$$\mathcal{D} := \text{clos} \begin{bmatrix} \mathcal{K}''_- \\ \mathcal{U}'\mathcal{K}'_+ \end{bmatrix} \subset \mathcal{H}_0, \quad \mathcal{D}_* := \text{clos} \begin{bmatrix} \mathcal{U}''^*\mathcal{K}''_- \\ \mathcal{K}'_+ \end{bmatrix} \subset \mathcal{H}_0. \quad (5.3)$$

Define an operator  $V: \mathcal{D} \rightarrow \mathcal{D}_*$  densely by

$$V = \begin{bmatrix} \mathcal{U}''^* & 0 \\ 0 & \mathcal{U}'^* \end{bmatrix} : \begin{bmatrix} k''_- \\ \mathcal{U}'k'_+ \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{U}''^*k''_- \\ k'_+ \end{bmatrix}. \quad (5.4)$$

for  $k''_- \in \mathcal{K}''_-$  and  $k'_+ \in \mathcal{K}'_+$ . By the same computation as in (4.9), we see that  $V$  extends to define an isometry from  $\mathcal{D}$  onto  $\mathcal{D}_*$ . Notice also that  $V$  is completely determined by the problem data.

Let us say that the operator  $\mathcal{U}^*$  on  $\mathcal{K}$  is a *minimal unitary extension* of  $V$  if  $\mathcal{U}^*$  is unitary on  $\mathcal{K}$  and there is an isometric embedding  $i_{\mathcal{H}_0}: \mathcal{H}_0 \rightarrow \mathcal{K}$  of  $\mathcal{H}_0$  into  $\mathcal{K}$  such that

$$i_{\mathcal{H}_0}V = \mathcal{U}^*i_{\mathcal{H}_0}|_{\mathcal{D}}. \quad (5.5)$$

and

$$\overline{\text{span}}_{n \in \mathbb{Z}} \mathcal{U}^n \text{im } i_{\mathcal{H}_0} = \mathcal{K}. \quad (5.6)$$

In this situation note that we then also have

$$i_{\mathcal{H}_0}V^* = \mathcal{U}i_{\mathcal{H}_0}|_{\mathcal{D}_*}. \quad (5.7)$$

Two such minimal unitary extensions  $(\mathcal{U}^*, i_{\mathcal{H}_0}; \mathcal{K})$  and  $(\tilde{\mathcal{U}}^*, \tilde{i}_{\mathcal{H}_0}; \tilde{\mathcal{K}})$  are said to be *unitarily equivalent* if there is a unitary operator  $\tau: \mathcal{K} \rightarrow \tilde{\mathcal{K}}$  such that

$$\tau\mathcal{U}^* = \tilde{\mathcal{U}}^*\tau, \quad \tau i_{\mathcal{H}_0} = \tilde{i}_{\mathcal{H}_0}.$$

Then the connection between minimal unitary extensions of  $V$  and lifts of  $X$  is given by the following.

**Theorem 5.1.** *Suppose that we are given data for a Lifting Problem 1.1 as above. Assume that the subspaces  $\mathcal{K}'_+$  and  $\mathcal{K}''_-$  are  $*$ -cyclic. Let  $V: \mathcal{D} \rightarrow \mathcal{D}_*$  be the isometry given by (5.4). Then there exists a canonical one-to-one correspondence between equivalence classes of minimal AA-unitary couplings  $(\mathcal{U}, i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K})$  of  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$  such that the contractive intertwiner  $Y = i_{\mathcal{K}''}^* i_{\mathcal{K}'}$  lifts  $X$  on the one hand and equivalence classes of minimal unitary extensions  $(\mathcal{U}^*, i_{\mathcal{H}_0}; \mathcal{K})$  of  $V$  on the other.*

*Specifically, if  $(\mathcal{U}, i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K})$  is a minimal AA-unitary coupling of  $(\mathcal{U}'', \mathcal{K}'')$  and  $(\mathcal{U}', \mathcal{K}')$  with associated contractive intertwiner  $Y = i_{\mathcal{K}''}^* i_{\mathcal{K}'}$  lifting  $X$ , then the mapping*

$$i_{\mathcal{H}_0} := \begin{bmatrix} i_{\mathcal{K}''} & i_{\mathcal{K}'} \end{bmatrix} \begin{bmatrix} \mathcal{K}''_- \\ \mathcal{K}'_+ \end{bmatrix}$$

*extends to an isometric embedding of  $\mathcal{H}_0$  into  $\mathcal{K}$  and*

$$(\mathcal{U}^*, i_{\mathcal{H}_0}; \mathcal{K})$$

*is a minimal unitary extension of  $V$ . Conversely, if  $(\mathcal{U}^*, i_{\mathcal{H}_0}; \mathcal{K})$  is a minimal unitary extension of  $V$  and if we define isometric embedding operators  $i_{\mathcal{K}'}: \mathcal{K}' \rightarrow \mathcal{K}$  and  $i_{\mathcal{K}''}: \mathcal{K}'' \rightarrow \mathcal{K}$  via the wave operator construction*

$$i_{\mathcal{K}'} k' = \text{s-lim}_{n \rightarrow \infty} \mathcal{U}^{*n} i_{\mathcal{H}_0} \begin{bmatrix} 0 \\ \mathcal{U}^n k' \end{bmatrix}, \quad i_{\mathcal{K}''} k'' = \text{s-lim}_{n \rightarrow \infty} \mathcal{U}^n i_{\mathcal{H}_0} \begin{bmatrix} \mathcal{U}^{''*n} k'' \\ 0 \end{bmatrix}$$

*defined initially only for*

$$k' \in \bigcup_{m \geq 0} \mathcal{U}'^{*m} \mathcal{K}'_+, \quad k'' \in \bigcup_{m \geq 0} \mathcal{U}''^m \mathcal{K}''_-,$$

*and then extended uniquely to all of  $\mathcal{K}'$  and  $\mathcal{K}''$  respectively by continuity, then the collection*

$$(\mathcal{U}, i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K})$$

*is a minimal AA-unitary coupling of  $(\mathcal{U}'', \mathcal{K}'')$  and  $(\mathcal{U}', \mathcal{K}')$  with associated contractive intertwiner  $Y = i_{\mathcal{K}''}^* i_{\mathcal{K}'}$  lifting  $X$ .*

*Proof.* Suppose that  $(\mathcal{U}, i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K})$  is a minimal AA-unitary coupling of  $(\mathcal{U}', \mathcal{K}')$  and  $(\mathcal{U}'', \mathcal{K}'')$ . Define the map

$$i_{\mathcal{H}_0}: \begin{bmatrix} k''_- \\ k'_+ \end{bmatrix} \mapsto i_{\mathcal{K}''} k''_- + i_{\mathcal{K}'} k'_+. \quad (5.8)$$

Since  $i_{\mathcal{K}'}$  and  $i_{\mathcal{K}''}$  are isometric then  $i_{\mathcal{H}_0}$  is isometric if and only if

$$\langle i_{\mathcal{K}''} k''_-, i_{\mathcal{K}'} k'_+ \rangle_{\mathcal{K}} = \langle k''_-, k'_+ \rangle_{\mathcal{H}_0} := \langle k''_-, X k'_+ \rangle_{\mathcal{K}''}.$$

This in turn means that the intertwiner  $Y = i_{\mathcal{K}''}^* i_{\mathcal{K}'}$  lifts  $X$ . Now,

$$\begin{aligned} i_{\mathcal{H}_0} V \begin{bmatrix} k''_- \\ \mathcal{U}' k'_+ \end{bmatrix} &= i_{\mathcal{H}_0} \begin{bmatrix} \mathcal{U}^{''*} k''_- \\ k'_+ \end{bmatrix} = i_{\mathcal{K}''} \mathcal{U}^{''*} k''_- + i_{\mathcal{K}'} k'_+ \\ &= \mathcal{U}^* (i_{\mathcal{K}''} k''_- + i_{\mathcal{K}'} \mathcal{U}' k'_+) = \mathcal{U}^* i_{\mathcal{H}_0} \begin{bmatrix} k''_- \\ \mathcal{U}' k'_+ \end{bmatrix}. \end{aligned}$$

This in turn means that (5.5) holds. Thus,  $\mathcal{U}^*$  on  $\mathcal{K}$  with embedding  $i_{\mathcal{H}_0}$  is a unitary extension of  $V$ . Since, by assumption,  $\mathcal{K}'_+$  is  $*$ -cyclic for  $\mathcal{U}'$  on  $\mathcal{K}'$ ,  $\mathcal{K}''_-$  is  $*$ -cyclic for  $\mathcal{U}''$  on  $\mathcal{K}''$ , and since the AA-unitary coupling  $(\mathcal{U}, i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K})$  is minimal, then

$$\overline{\text{span}}_{n \in \mathbb{Z}} \mathcal{U}^n \text{im } i_{\mathcal{H}_0} = \mathcal{K}.$$

Thus,  $(\mathcal{U}^*, i_{\mathcal{H}_0}; \mathcal{K})$  is a minimal unitary extension of  $V$ .

Conversely, suppose that  $(\mathcal{U}^*, i_{\mathcal{H}_0}; \mathcal{K})$  is a minimal unitary extension of  $V$ . We now apply the construction of the wave operator from [45] (Section 4), which simplifies significantly in our situation due to the fact that  $\mathcal{K}'_+$  and  $\mathcal{K}''_-$  are embedded isometrically into  $\mathcal{H}_0$ . For  $k' \in \mathcal{U}'^{*n}\mathcal{K}'_+$  and  $m \geq n$  note that  $\mathcal{U}^{*m}i_{\mathcal{H}_0} \begin{bmatrix} 0 \\ \mathcal{U}'^m k' \end{bmatrix}$  is well-defined (since  $\mathcal{U}'^m k' \in \mathcal{K}'_+$  so  $\begin{bmatrix} 0 \\ \mathcal{U}'^m k' \end{bmatrix} \in \mathcal{H}_0$ ) and independent of  $m$  (since  $\mathcal{U}^*$  is an extension of  $V$ , see (5.7)). Thus, the formula

$$i_{\mathcal{K}'} : k' \mapsto \lim_{m \rightarrow \infty} \mathcal{U}^{*m} i_{\mathcal{H}_0} \begin{bmatrix} 0 \\ \mathcal{U}'^m k' \end{bmatrix} \quad (5.9)$$

is a well-defined isometry from  $\bigcup_{n=0}^{\infty} \mathcal{U}'^{*n}\mathcal{K}'_+$  into  $\mathcal{K}$ . By assumption,  $\bigcup_{n=0}^{\infty} \mathcal{U}'^{*n}\mathcal{K}'_+$  is dense in  $\mathcal{K}'$ , and hence  $i_{\mathcal{K}'}$  extends uniquely by continuity to an isometry (still denoted by  $i_{\mathcal{K}'}$ ) from  $\mathcal{K}'$  into  $\mathcal{K}$ . Similarly, the formula

$$i_{\mathcal{K}''} : k'' \mapsto \lim_{m \rightarrow \infty} \mathcal{U}^m i_{\mathcal{H}_0} \begin{bmatrix} \mathcal{U}''^{*m} k'' \\ 0 \end{bmatrix} \quad (5.10)$$

gives rise to a well-defined isometry from  $\mathcal{K}''$  into  $\mathcal{K}$ . Definitions (5.9) and (5.10) imply that

$$\mathcal{U} i_{\mathcal{K}'} = i_{\mathcal{K}'} \mathcal{U}' \quad \text{and} \quad \mathcal{U} i_{\mathcal{K}''} = i_{\mathcal{K}''} \mathcal{U}''.$$

We have thus arrived at an AA-unitary coupling  $(\mathcal{U}, i_{\mathcal{K}'}, i_{\mathcal{K}''}; \mathcal{K})$  of  $(\mathcal{U}'', \mathcal{K}'')$  and  $(\mathcal{U}', \mathcal{K}')$ . To check the minimality of the AA-unitary coupling note that it follows from (5.9) and (5.10) that

$$\text{im } i_{\mathcal{K}'} = \overline{\text{span}}_{n \in \mathbb{Z}_-} \mathcal{U}^n i_{\mathcal{H}_0} \begin{bmatrix} 0 \\ \mathcal{K}'_+ \end{bmatrix}$$

and

$$\text{im } i_{\mathcal{K}''} = \overline{\text{span}}_{n \in \mathbb{Z}_+} \mathcal{U}^n i_{\mathcal{H}_0} \begin{bmatrix} \mathcal{K}''_- \\ 0 \end{bmatrix}.$$

Since  $i_{\mathcal{H}_0} \begin{bmatrix} 0 \\ \mathcal{K}'_+ \end{bmatrix}$  is invariant for  $\mathcal{U}$  and  $i_{\mathcal{H}_0} \begin{bmatrix} \mathcal{K}''_- \\ 0 \end{bmatrix}$  is invariant for  $\mathcal{U}^*$ , we conclude that

$$\text{im } i_{\mathcal{K}'} + \text{im } i_{\mathcal{K}''} \supseteq \text{span}_{n \in \mathbb{Z}} \mathcal{U}^n i_{\mathcal{H}_0} \begin{bmatrix} \mathcal{K}''_- \\ \mathcal{K}'_+ \end{bmatrix}.$$

Therefore,

$$\overline{\text{im } i_{\mathcal{K}'} + \text{im } i_{\mathcal{K}''}} \supseteq \overline{\text{span}}_{n \in \mathbb{Z}} \mathcal{U}^n i_{\mathcal{H}_0} = \mathcal{K}.$$

The last equality is due to minimality of the extension. Thus,

$$\overline{\text{im } i_{\mathcal{K}'} + \text{im } i_{\mathcal{K}''}} = \mathcal{K}$$

and it follows that the AA-unitary coupling is minimal. Moreover,  $Y = i_{\mathcal{K}''}^* i_{\mathcal{K}'}$  lifts  $X$  since

$$\langle Y k'_+, k''_- \rangle_{\mathcal{K}} = \langle i_{\mathcal{K}'} k'_+, i_{\mathcal{K}''} k''_- \rangle_{\mathcal{K}} = \left\langle \begin{bmatrix} 0 \\ k'_+ \end{bmatrix}, \begin{bmatrix} k''_- \\ 0 \end{bmatrix} \right\rangle_{\mathcal{H}_0} = \langle X k'_+, k''_- \rangle_{\mathcal{K}''}.$$

The correspondences between AA-unitary couplings and unitary extensions defined above are mutually inverse. Moreover, it is straightforward from the definitions of the equivalences that under these correspondences equivalent AA-unitary couplings go to equivalent unitary extensions and equivalent unitary extensions go to equivalent AA-unitary couplings. This completes the proof of Theorem 5.1.  $\square$



## 6. STRUCTURE OF UNITARY EXTENSIONS

In the previous section we obtained a correspondence between contractive intertwining lifts  $Y$  of  $X$  and minimal unitary extensions  $\mathcal{U}^*$  of a isometry  $V$  on a Hilbert space  $\mathcal{H}_0$  with domain  $\mathcal{D}$  and codomain  $\mathcal{D}_*$ . In this section we indicate how one can parametrize all such minimal unitary extensions.

We therefore suppose that we are given a Hilbert space  $\mathcal{H}_0$ , two subspaces  $\mathcal{D}$  and  $\mathcal{D}_*$  of  $\mathcal{H}_0$  and an operator  $V$  which maps  $\mathcal{D}$  isometrically onto  $\mathcal{D}_*$ :

$$V: \mathcal{D} \rightarrow \mathcal{D}_*.$$

In this situation we say that  $V$  is an *isometry* on  $\mathcal{H}_0$  with *domain*  $\mathcal{D}$  and *codomain*  $\mathcal{D}_*$ . We let  $\Delta$  and  $\Delta_*$  be the respective orthogonal complements

$$\Delta := \mathcal{H}_0 \ominus \mathcal{D}, \quad \Delta_* := \mathcal{H}_0 \ominus \mathcal{D}_*.$$

Let  $\mathcal{U}^*$  be a minimal unitary extension of  $V$  to a Hilbert space  $\mathcal{K}$ , i.e.,  $\mathcal{U}$  is unitary on the Hilbert space  $\mathcal{K}$ ,  $\mathcal{K}$  contains the space  $\mathcal{H}_0$  as a subspace, the smallest subspace of  $\mathcal{K}$  containing  $\mathcal{H}_0$  and reducing for  $\mathcal{U}$  is the whole space  $\mathcal{K}$  and  $\mathcal{U}^*$  when restricted to  $\mathcal{D} \subset \mathcal{H}_0 \subset \mathcal{K}$  agrees with  $V$ :  $\mathcal{U}|_{\mathcal{D}} = V$ . We set  $\mathcal{H}_1$  equal to  $\mathcal{K} \ominus \mathcal{H}_0$  and write  $\mathcal{K} = \mathcal{H}_0 \oplus \mathcal{H}_1$ . We associate two unitary colligations  $U_1$  and  $U_0$  to the extension  $\mathcal{U}^*$  as follows. Since  $\mathcal{U}^*|_{\mathcal{D}} = V$  maps  $\mathcal{D}$  onto  $\mathcal{D}_*$  and since  $\mathcal{U}^*$  is unitary, necessarily  $\mathcal{U}^*$  must map  $\mathcal{K} \ominus \mathcal{D} = \Delta \oplus \mathcal{H}_1$  onto  $\mathcal{K} \ominus \mathcal{D}_* = \Delta_* \oplus \mathcal{H}_1$ . To define the unitary colligation  $U_1$ , we introduce a second copy  $\tilde{\Delta}$  of  $\Delta$  and a second copy  $\tilde{\Delta}_*$  of  $\Delta_*$  together with unitary identification maps

$$i_{\tilde{\Delta}}: \tilde{\Delta} \rightarrow \Delta \subset \mathcal{H}_0 \subset \mathcal{K}, \quad i_{\tilde{\Delta}_*}: \tilde{\Delta}_* \rightarrow \Delta_* \subset \mathcal{H}_0 \subset \mathcal{K}. \quad (6.1)$$

We then define the colligation

$$U_1 := \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}: \begin{bmatrix} \mathcal{H}_1 \\ \tilde{\Delta} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \tilde{\Delta}_* \end{bmatrix} \quad (6.2)$$

by

$$U_1 = \begin{bmatrix} i_{\mathcal{H}_1}^* \\ i_{\tilde{\Delta}_*}^* \end{bmatrix} \mathcal{U}^* \begin{bmatrix} i_{\mathcal{H}_1} & i_{\tilde{\Delta}} \end{bmatrix} \quad (6.3)$$

where  $i_{\mathcal{H}_1}: \mathcal{H}_1 \rightarrow \mathcal{K} = \mathcal{H}_0 \oplus \mathcal{H}_1$  is the natural inclusion map. We define a second colligation

$$U_0 = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}: \begin{bmatrix} \mathcal{H}_0 \\ \tilde{\Delta}_* \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_0 \\ \tilde{\Delta} \end{bmatrix}$$

as follows. Note that the space  $\mathcal{H}_0$  has two orthogonal decompositions

$$\mathcal{H}_0 = \mathcal{D} \oplus \Delta = \mathcal{D}_* \oplus \Delta_*.$$

If we use the first orthogonal decomposition of  $\mathcal{H}_0$  on the domain side and the second orthogonal decomposition of  $\mathcal{H}_0$  on the range side, then we may define an operator  $U_0: \mathcal{H}_0 \oplus \tilde{\Delta}_* \rightarrow \mathcal{H}_0 \oplus \tilde{\Delta}$  via the  $3 \times 3$ -block matrix

$$U_0 = \begin{bmatrix} V & 0 & 0 \\ 0 & 0 & i_{\tilde{\Delta}_*} \\ 0 & i_{\tilde{\Delta}}^* & 0 \end{bmatrix}: \begin{bmatrix} \mathcal{D} \\ \Delta \\ \tilde{\Delta}_* \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{D}_* \\ \Delta_* \\ \tilde{\Delta} \end{bmatrix}, \quad (6.4)$$

or, in colligation form,

$$U_0 = \begin{bmatrix} A_0 & B_0 \\ C_0 & 0 \end{bmatrix}: \begin{bmatrix} \mathcal{H}_0 \\ \tilde{\Delta}_* \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_0 \\ \tilde{\Delta} \end{bmatrix} \quad (6.5)$$

where

$$\begin{aligned} A_0|_{\mathcal{D}} &= V, \quad A_0|_{\Delta} = 0, \quad C_0|_{\mathcal{D}} = 0, \quad C_0|_{\Delta} = i_{\Delta}^*, \\ B_0 &= i_{\tilde{\Delta}^*} \text{ with } \text{im } B_0 = \Delta_* \subset \mathcal{H}_0. \end{aligned} \quad (6.6)$$

We note that the colligation  $U_0$  is defined by the problem data (i.e., the isometry  $V$  with given domain  $\mathcal{D}$  and codomain  $\mathcal{D}_*$  in the space  $\mathcal{H}_0$ ) and is independent of the choice of unitary extension  $\mathcal{U}^*$ . As one sweeps all possible unitary extensions  $\mathcal{U}^*$  of  $V$ , the associated colligation  $U_1$  can be an arbitrary colligation of the form (6.2), i.e., one with input space  $\tilde{\Delta}$  and output space  $\tilde{\Delta}_*$ . Moreover, from the fact the colligation matrix  $U_0 = \begin{bmatrix} A_0 & B_0 \\ C_0 & 0 \end{bmatrix}$  has a zero for its  $(2,2)$ -entry, we see from Theorem 2.1 that the feedback connection  $\mathcal{F}_\ell(U_0, U_1)$  is well-defined for *any* colligation (in particular, for any unitary colligation) of the form (6.2). Also, from the very definitions, we see that if  $U_1$  is constructed from the unitary extension  $\mathcal{U}^*$  as indicated in (6.3), then we recover  $\mathcal{U}^*$  from  $U_0$  and  $U_1$  as the feedback connection  $\mathcal{U}^* = \mathcal{F}_\ell(U_0, U_1)$  given by (2.2). The following result gives the converse.

**Theorem 6.1.** *The operator  $\mathcal{U}^*$  on  $\mathcal{K}$  is a unitary extension of  $V$  to a Hilbert space  $\mathcal{K}$  if and only if, upon decomposing  $\mathcal{K}$  as  $\mathcal{K} = \mathcal{H}_0 \oplus \mathcal{H}_1$ ,  $\mathcal{U}^*$  can be written in the form*

$$\mathcal{U}^* = \mathcal{F}_\ell(U_0, U_1)$$

where  $U_0$  is the universal unitary colligation determined completely by the problem data as in (6.6) and  $U_1$  is a free-parameter unitary colligation of the form (6.2). Moreover,  $\mathcal{U}^*$  is a minimal unitary extension of  $V$ , i.e., the smallest reducing subspace for  $\mathcal{U}^*$  containing  $\mathcal{H}_0$  is the whole space  $\mathcal{K} := \mathcal{H}_0 \oplus \mathcal{H}_1$ , if and only if  $U_1$  is a simple unitary colligation, i.e., the smallest reducing subspace for  $A_1$  containing  $\text{im } B_1 + \text{im } C_1^*$  is the whole space  $\mathcal{H}_1$ .

*Proof.* We already showed that every unitary extension  $\mathcal{U}^*$  of  $V$  has the form  $\mathcal{U}^* = \mathcal{F}_\ell(U_0, U_1)$  where  $\mathcal{U}^*$  determines  $U_1$  according to (6.3). Conversely we now show that every lower feedback connection  $\mathcal{F}_\ell(U_0, U_1)$  (with arbitrary unitary colligation  $U_1$  of the form (6.2)) produces a unitary extension  $\mathcal{U}^*$  of  $V$ . From the formula (2.4) for the lower feedback connection applied to the case where  $D_0 = 0$ , we see that

$$\mathcal{F}_\ell(U_0, U_1) \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} A_0 + B_0 D_1 C_0 & B_0 C_1 \\ B_1 C_0 & A_1 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix}. \quad (6.7)$$

Specializing to the case where  $h_0 = d \in \mathcal{D} \subset \mathcal{H}_0$  and  $h_1 = 0$  and using the formulas (6.6) for  $A_0, B_0, C_0$ , we see that

$$\mathcal{F}_\ell(U_0, U_1) \begin{bmatrix} d \\ 0 \end{bmatrix} = \begin{bmatrix} A_0 d \\ 0 \end{bmatrix} = \begin{bmatrix} Vd \\ 0 \end{bmatrix}$$

and it follows that  $\mathcal{F}_\ell(U_0, U_1)$  is an extension of  $V$ . Moreover, by plugging in the explicit formulas (6.6) for  $A_0, B_0, C_0$  into (6.7), it is straightforward to verify that we recover  $U_1$  from  $\mathcal{U}^* := \mathcal{F}_\ell(U_0, U_1)$  via the formula (6.3) and that  $\mathcal{U}^*$  is unitary exactly when  $U_1$  is unitary.

It remains to check:  $\mathcal{U}^*$  is a minimal extension of  $V$  if and only if  $U_1$  is a simple unitary colligation. Consider the minimal reducing subspace for  $\mathcal{U}^*$  that contains  $\mathcal{H}_0$ , then its orthogonal complement (which is a subspace of  $\mathcal{H}_1$ ) also reduces  $\mathcal{U}^*$ . From the definitions one sees that the latter is the zero subspace exactly when  $U_1$  is simple.  $\square$

Since unitary extensions  $\mathcal{U}^*$  of  $V$  are given via the lower feedback connection  $\mathcal{F}_\ell(U_0, U_1)$ , we may use the results of Theorems 2.3 and 2.4 to compute positive and negative powers of  $\mathcal{U}^*$ . To simplify notation, we let

$$\begin{aligned}\mathbf{S}^+ &= \begin{bmatrix} S_0^+ & S_2^+ \\ S_1^+ & S^+ \end{bmatrix} : \begin{bmatrix} \mathcal{H}_0 \\ \ell_{\tilde{\Delta}^*}(\mathbb{Z}_+) \end{bmatrix} \rightarrow \begin{bmatrix} \ell_{\mathcal{H}_0}(\mathbb{Z}_+) \\ \ell_{\tilde{\Delta}}(\mathbb{Z}_+) \end{bmatrix}, \\ \mathbf{S}^- &= \begin{bmatrix} S_0^- & S_1^- \\ S_2^- & S^- \end{bmatrix} : \begin{bmatrix} \mathcal{H}_0 \\ \ell_{\tilde{\Delta}}(\mathbb{Z}_-) \end{bmatrix} \rightarrow \begin{bmatrix} \ell_{\mathcal{H}_0}(\mathbb{Z}_-) \\ \ell_{\tilde{\Delta}^*}(\mathbb{Z}_-) \end{bmatrix}\end{aligned}\quad (6.8)$$

be the forward and backward augmented input-output operators for the universal colligation  $U_0$  and we let

$$\begin{aligned}\mathbf{\Omega}^+ &= \begin{bmatrix} \Omega_0^+ & \Omega_2^+ \\ \Omega_1^+ & \Omega^+ \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \ell_{\tilde{\Delta}}(\mathbb{Z}_+) \end{bmatrix} \rightarrow \begin{bmatrix} \ell_{\mathcal{H}_1}(\mathbb{Z}_+) \\ \ell_{\tilde{\Delta}^*}(\mathbb{Z}_+) \end{bmatrix}, \\ \mathbf{\Omega}^- &= \begin{bmatrix} \Omega_0^- & \Omega_1^- \\ \Omega_2^- & \Omega^- \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \ell_{\tilde{\Delta}^*}(\mathbb{Z}_-) \end{bmatrix} \rightarrow \begin{bmatrix} \ell_{\mathcal{H}_1}(\mathbb{Z}_-) \\ \ell_{\tilde{\Delta}}(\mathbb{Z}_-) \end{bmatrix}\end{aligned}\quad (6.9)$$

be the forward and backward augmented input-output operators for the free-parameter unitary colligation  $U_1$ . From the first rows in the formulas (2.22) and (2.28), we read off that

$$\Lambda_{\mathcal{H}_0}(U) = \begin{bmatrix} \Lambda_{\mathcal{H}_0,-}(U) \\ \Lambda_{\mathcal{H}_0,+}(U) \end{bmatrix} : \begin{bmatrix} \mathcal{H}_0 \\ \mathcal{H}_1 \end{bmatrix} \rightarrow \begin{bmatrix} \ell_{\mathcal{H}_0}(\mathbb{Z}_-) \\ \ell_{\mathcal{H}_0}(\mathbb{Z}_+) \end{bmatrix}$$

is given by

$$\Lambda_{\mathcal{H}_0}(U) = \begin{bmatrix} S_0^- + S_1^-(I - \Omega^- S^-)^{-1} \Omega^- S_2^- & S_1^-(I - \Omega^- S^-)^{-1} \Omega_2^- \\ S_0^+ + S_2^+(I - \Omega^+ S^+)^{-1} \Omega^+ S_1^+ & S_2^+(I - \Omega^+ S^+)^{-1} \Omega_1^+ \end{bmatrix}. \quad (6.10)$$

From the second rows in the formulas (2.22) and (2.28) we read off that

$$\Lambda_{\mathcal{H}_1}(U) = \begin{bmatrix} \Lambda_{\mathcal{H}_1,-}(U) \\ \Lambda_{\mathcal{H}_1,+}(U) \end{bmatrix} : \begin{bmatrix} \mathcal{H}_0 \\ \mathcal{H}_1 \end{bmatrix} \rightarrow \begin{bmatrix} \ell_{\mathcal{H}_1}(\mathbb{Z}_-) \\ \ell_{\mathcal{H}_1}(\mathbb{Z}_+) \end{bmatrix}$$

is given by

$$\Lambda_{\mathcal{H}_1}(U) = \begin{bmatrix} \Omega_1^-(I - S^- \Omega^-)^{-1} S_2^- & \Omega_0^- + \Omega_1^-(I - S^- \Omega^-)^{-1} S^- \Omega_2^- \\ \Omega_2^+(I - S^+ \Omega^+)^{-1} S_1^+ & \Omega_0^+ + \Omega_2^+(I - S^+ \Omega^+)^{-1} S^+ \Omega_1^+ \end{bmatrix}. \quad (6.11)$$

## 7. PARAMETRIZATION OF SYMBOLS OF INTERTWINERS

Assume that we are given the data set

$$(X, (\mathcal{U}', \mathcal{K}'), (\mathcal{U}'', \mathcal{K}''), \mathcal{K}'_+ \subset \mathcal{K}', \mathcal{K}'' \subset \mathcal{K}'') \quad (7.1)$$

as in the Lifting Problem 1.1. If we are given  $*$ -cyclic subspaces  $\mathcal{G}'$  and  $\mathcal{G}''$  for  $\mathcal{U}'$  and  $\mathcal{U}''$  respectively, then the sets

$$\{\mathcal{U}'^n g' : n \in \mathbb{Z}, g' \in \mathcal{G}'\}, \quad \{\mathcal{U}''^n g'' : n \in \mathbb{Z}, g'' \in \mathcal{G}''\}$$

have dense span in  $\mathcal{K}'$  and  $\mathcal{K}''$  respectively. If  $Y \in \mathcal{L}(\mathcal{K}', \mathcal{K}'')$  is any operator satisfying the intertwining condition  $Y\mathcal{U}' = \mathcal{U}''Y$ , then the computation

$$\langle Y\mathcal{U}'^n g', \mathcal{U}''^m g'' \rangle_{\mathcal{K}''} = \langle \mathcal{U}''^{(n-m)} Y g', g'' \rangle_{\mathcal{K}''} = \langle i_{\mathcal{G}''}^* \mathcal{U}''^{(n-m)} Y i_{\mathcal{G}'} g', g'' \rangle_{\mathcal{G}''}$$

shows that  $Y$  is uniquely determined by its *symbol*

$$\{Y_n = i_{\mathcal{G}''}^* \mathcal{U}''^{*n} Y i_{\mathcal{G}'} = i_{\mathcal{G}''}^* Y \mathcal{U}'^{*n} i_{\mathcal{G}'}\}_{n \in \mathbb{Z}}.$$

Therefore, in principle, to describe all contractive intertwining lifts  $Y$  of a given  $X: \mathcal{K}'_+ \rightarrow \mathcal{K}''_-$ , it suffices to describe all the symbols  $w_Y$  of contractive intertwining lifts  $Y$ . Such a description is given in the next result.

**Theorem 7.1.** *Suppose that we are given data set (7.1) for a Lifting Problem 1.1. Let  $U_0: \begin{bmatrix} \mathcal{H}_0 \\ \tilde{\Delta}_* \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_0 \\ \tilde{\Delta} \end{bmatrix}$  be the universal unitary colligation constructed from the problem data as in (6.4) or (6.5) and (6.6) with associated augmented input-output maps  $\mathbf{S}^+$  and  $\mathbf{S}^-$  as in (6.8). For  $U_1$  equal to a free-parameter unitary colligation of the form (6.2), let  $\Omega^+$  and  $\Omega^-$  be the associated augmented input-output maps as in (6.9). Finally let  $\mathcal{G}'$  and  $\mathcal{G}''$  be a fixed pair of  $\mathcal{U}'$ -\*-cyclic and  $\mathcal{U}''$ -\*-cyclic subspaces of  $\mathcal{K}'$  and  $\mathcal{K}''$  and assume that*

$$\mathcal{G}' \subset \mathcal{K}'_+, \quad \mathcal{G}'' \subset \mathcal{K}''_-.$$

Let  $i_{\mathcal{G}'}: \mathcal{G}' \rightarrow \mathcal{H}_0$  be the inclusion map of  $\mathcal{G}'$  into  $\mathcal{H}_0$  obtained as the inclusion of  $\mathcal{G}'$  in  $\mathcal{K}'_+$  followed by the inclusion of  $\mathcal{K}'_+$  into  $\mathcal{H}_0$ , and, similarly, let  $i_{\mathcal{G}''}$  be the inclusion of  $\mathcal{G}''$  in  $\mathcal{H}_0$  obtained as the inclusion of  $\mathcal{G}''$  in  $\mathcal{K}''_-$  followed by the inclusion of  $\mathcal{K}''_-$  in  $\mathcal{H}_0$ . Let  $\mathcal{I}_{\mathcal{G}''}^* = \text{diag}_{n \in \mathbb{Z}} \{i_{\mathcal{G}''}^*\}$  be the coordinate-wise projection of  $\ell_{\mathcal{H}_0}(\mathbb{Z})$  onto  $\ell_{\mathcal{G}''}(\mathbb{Z})$ . Then the  $\mathcal{L}(\mathcal{G}', \mathcal{G}'')$ -valued bilateral sequence  $w = \{w(n)\}_{n \in \mathbb{Z}}$  is the symbol  $w = w_Y$  (with respect to  $\mathcal{G}'$  and  $\mathcal{G}''$ ) for a contractive intertwining lift  $Y$  of  $X$  if and only if there exists a free-parameter unitary colligation  $U_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}: \mathcal{H}_1 \oplus \tilde{\Delta} \rightarrow \mathcal{H}_1 \oplus \tilde{\Delta}_*$  so that  $w$  (as an infinite column vector) has the form

$$w_Y = \begin{bmatrix} \mathcal{I}_{\mathcal{G}''}^* S_0^- i_{\mathcal{G}'} \\ \mathcal{I}_{\mathcal{G}''}^* [S_0^+ + S_2^+ (I - \Omega^+ S^+)^{-1} \Omega^+ S_1^+] i_{\mathcal{G}'} \end{bmatrix}. \quad (7.2)$$

**Remark 7.2.** There are various other formulations of the formula (7.2) for the parametrization of lifting symbols. If we define

$$s_0^+ = \mathcal{I}_{\mathcal{G}''}^* S_0^+ i_{\mathcal{G}'}, \quad s_2^+ = \mathcal{I}_{\mathcal{G}''}^* S_2^+, \quad s_1^+ = S_1^+ i_{\mathcal{G}'}, \quad s^+ = S^+, \quad (7.3)$$

then the formula (7.2) assumes the form

$$w_Y = \begin{bmatrix} s_0^+ & s_2^+ \\ s_1^+ & s^+ \end{bmatrix} \begin{bmatrix} s_0^- \\ s_2^- \end{bmatrix} \begin{bmatrix} I - \Omega^+ s^+ \end{bmatrix}^{-1} \Omega^+ s_1^+ \quad (7.4)$$

where the coefficient matrix  $\begin{bmatrix} s_0^+ & s_2^+ \\ s_1^+ & s^+ \end{bmatrix}$  together with  $s_0^-$  is completely determined from the problem data while  $\Omega^+$  is the input-output map for the free-parameter unitary colligation  $U_1$ .

If we consider  $\ell_{\mathcal{G}''}(\mathbb{Z}_+)$  as embedded in  $\ell_{\mathcal{G}''}(\mathbb{Z})$  in the natural way, we may rewrite in turn the formula (7.4) in the still more compact form

$$w_Y = s_0 + s_2(I - \omega s)^{-1} \omega s_1 \quad (7.5)$$

where we define

$$\begin{aligned} (s_0 g')(m) &= \begin{cases} (s_0^- g')(m) & \text{for } m < 0 \\ (s_0^+ g')(m) & \text{for } m \geq 0, \end{cases} \\ s_2 &= \iota s_2^+ \text{ where } \iota: \ell_{\mathcal{G}''}(\mathbb{Z}_+) \rightarrow \ell_{\mathcal{G}''}(\mathbb{Z}) \text{ is the natural inclusion,} \\ s &= s^+, \quad \omega = \Omega^+, \quad s_1 = s_1^+. \end{aligned} \quad (7.6)$$

*Proof.* Theorem 5.1 gives an identification between contractive intertwining lifts  $\mathcal{U}^*$  and unitary extensions of the isometry  $V: \mathcal{D} \rightarrow \mathcal{D}_*$  on  $\mathcal{H}_0$  constructed from the

Lifting Problem data while Theorem 6.1 in turn gives a Redheffer-type parametrization of all such unitary extensions. Moreover formula (6.10) tells us how to compute the powers of  $\mathcal{U}^* = \mathcal{F}_\ell(U_0, U_1)$  followed by the projection to the subspace  $\mathcal{H}_0$ . By definition the symbol  $w_Y$  is given by

$$w_Y(n) = i_{\mathcal{G}''}^* \mathcal{U}^{*n} i_{\mathcal{G}}.$$

The parametrization result (7.2) now follows by plugging into (6.10) once we verify:

$$i_{\mathcal{G}''}^* S_1^- (I - \Omega^- S^-)^{-1} S_2^- i_{\mathcal{G}'} = 0. \quad (7.7)$$

We assert that in fact

$$S_2^- i_{\mathcal{G}'} = 0. \quad (7.8)$$

Indeed, by definition  $S_2^- h_0 = \{\tilde{\delta}_*(n)\}_{n \in \mathbb{Z}_-}$  means that  $\tilde{\delta}_*(n)$  is generated by the recursion

$$\begin{aligned} h_0(n) &= A_0^* h_0(n+1), \quad h_0(0) = h_0, \\ \tilde{\delta}_*(n) &= B_0^* h_0(n) \text{ for } n = -1, -2, \dots \end{aligned}$$

If we set  $m = -n$ , this means simply that

$$\tilde{\delta}_*(-m) = B_0^* A_0^{*m} h_0.$$

For the case where  $h_0 = i_{\mathcal{G}'} g' \in i_{\mathcal{K}'_+} \mathcal{K}'_+ \subset \mathcal{R}$ , we then have

$$A_0^* h_0 = V^* i_{\mathcal{G}'} g' = i_{\mathcal{K}'_+} \mathcal{U}' g' \in \mathcal{D}_*$$

and, inductively, given that  $A_0^{*m} h_0 = i_{\mathcal{K}'_+} k'_+ \in i_{\mathcal{K}'_+} \mathcal{K}'_+ \subset \mathcal{D}_*$ , we have

$$A_0^{*m+1} h_0 = A_0^* i_{\mathcal{K}'_+} k'_+ = V^* i_{\mathcal{K}'_+} k'_+ = i_{\mathcal{K}'_+} \mathcal{U}' k'_+ \in i_{\mathcal{K}'_+} \mathcal{K}'_+ \subset \mathcal{D}_*.$$

As  $\mathcal{D}_*$  is orthogonal to the final space  $\Delta_*$  for the isometry  $i_{\tilde{\Delta}_*}$ , it follows that, for  $m = 1, 2, \dots$ ,

$$B_0^* A_0^{*m} i_{\mathcal{G}'} g' = i_{\tilde{\Delta}_*}^* A_0^{*m} i_{\mathcal{G}'} g' = 0 \text{ for } m = 1, 2, \dots$$

from which (7.8) and (7.7) follow.  $\square$

**Remark 7.3.** We note that the value of the symbol  $w_Y(n)$  is independent of the choice of lift  $Y$  for  $n \leq 0$ . Indeed, for  $n \leq 0$ ,  $g' \in \mathcal{G}'$  and  $g'' \in \mathcal{G}''$  (where as always we are assuming that  $\mathcal{G}' \subset \mathcal{K}'_+$  and  $\mathcal{G}'' \subset \mathcal{K}''_-$ ), we have

$$\begin{aligned} \langle w_Y(n) g', g'' \rangle_{\mathcal{G}''} &= \langle \mathcal{U}^{''*n} Y g', g'' \rangle_{\mathcal{K}''_-} = \langle Y g', \mathcal{U}^{''n} g'' \rangle_{\mathcal{K}''_-} \\ &= \langle Y g', \mathcal{U}^{''n} g'' \rangle_{\mathcal{K}''_-} = \langle X g', \mathcal{U}^{''n} g'' \rangle_{\mathcal{K}''_-} \end{aligned}$$

since  $\mathcal{U}^{''n} g'' \in \mathcal{K}''_-$  for  $n \leq 0$  whenever  $g'' \in \mathcal{G}'' \subset \mathcal{K}''_-$ .

Let us consider the special case where we take

$$\mathcal{G}' := \mathcal{K}'_+ \text{ and } \mathcal{G}'' := \mathcal{E}'' := \mathcal{K}''_- \ominus \mathcal{U}^{''*} \mathcal{K}''_-.$$

Then  $\mathcal{E}''$  is wandering for  $\mathcal{U}''$  and we may represent  $\mathcal{K}''$  as the direct-sum decomposition

$$\mathcal{K}'' = \mathcal{K}''_- \oplus \bigoplus_{n=0}^{\infty} \mathcal{U}^{''n} (\mathcal{U}'' \mathcal{E}'').$$

Then the Fourier representation operator

$$\Phi'': k'' \mapsto \{i_{\mathcal{E}''}^* \mathcal{U}^{''*n+1} k''\}_{n \in \mathbb{Z}_+}$$

is a coisometry mapping  $\mathcal{K}''$  onto  $\ell_{\mathcal{E}''}^2(\mathbb{Z}_+)$  with initial space equal to  $\mathcal{K}'' \ominus \mathcal{K}_-'' = \bigoplus_{n=0}^{\infty} \mathcal{U}''^n(\mathcal{U}''\mathcal{E}'')$ .

If  $Y$  is any lift, then  $Y$  is uniquely determined by its restriction  $Y|_{\mathcal{K}'_+}$  to  $\mathcal{K}'_+$  by the wave-operator construction; thus, to solve the Lifting Problem it suffices to describe all  $Y|_{\mathcal{K}'_+} : \mathcal{K}'_+ \rightarrow \mathcal{K}''$  rather than all lifts  $Y : \mathcal{K}' \rightarrow \mathcal{K}''$ . Moreover, if we use the Fourier representation operator  $\Phi''$  to identify  $\mathcal{K}'' \ominus \mathcal{K}_-''$  with  $\ell_{\mathcal{E}''}^2(\mathbb{Z}_+)$ , then we have an identification of  $\mathcal{K}''$  with  $\begin{bmatrix} \mathcal{K}_-'' \\ \ell_{\mathcal{E}''}^2(\mathbb{Z}_+) \end{bmatrix}$ . Then, with this identification in place, the restricted lift  $Y|_{\mathcal{K}'_+}$  has a matrix representation of the form

$$Y|_{\mathcal{K}'_+} = \begin{bmatrix} X \\ Y_+ \end{bmatrix} : \mathcal{K}'_+ \rightarrow \begin{bmatrix} \mathcal{K}_-'' \\ \ell_{\mathcal{E}''}^2(\mathbb{Z}_+) \end{bmatrix}.$$

With this representation we lose no information concerning the lift  $Y$  despite the fact that in general  $\mathcal{G}'' := \mathcal{E}'' \subset \mathcal{K}_-''$  may not be  $*$ -cyclic for  $\mathcal{U}''$ .

If we use the parametrization from (7.2), the operator  $Y_+$  in turn has an infinite column-matrix representation given by

$$Y_+ = \begin{bmatrix} w_Y(1) \\ w_Y(2) \\ \vdots \\ w_Y(n+1) \\ \vdots \end{bmatrix} = J_+^* \mathcal{I}_{\mathcal{E}''}^* (S_0^+ + S_2^+ (I - \Omega^+ S^+)^{-1} \Omega^+ S_1^+) i_{\mathcal{K}'_+}$$

where  $J_+$  is the shift operator on  $\ell_{\mathcal{E}''}^2(\mathbb{Z}_+)$  and where  $\Omega^+$  is the input-output map associated with the free-parameter unitary colligation  $U_1$ . Finally, if we apply the  $Z$ -transform

$$\{e''(n)\}_{n \in \mathbb{Z}_+} \mapsto \sum_{n=0}^{\infty} e''(n) \zeta^n$$

to transform  $\ell_{\mathcal{E}''}^2(\mathbb{Z}_+)$  to the Hardy space  $H_{\mathcal{E}''}^2$ , then the operator  $\widehat{Y}_+ : \mathcal{K}'_+ \rightarrow H_{\mathcal{E}''}^2$  induced by  $Y_+ : \mathcal{K}'_+ \rightarrow \ell_{\mathcal{E}''}^2(\mathbb{Z}_+)$  is given by multiplication by the  $\mathcal{L}(\mathcal{K}'_+, \mathcal{E}'')$ -valued function

$$\widehat{Y}_+(\zeta) = \zeta^{-1} [\widehat{s}_0^+(\zeta) - \widehat{s}_0^+(0)] + \zeta^{-1} \widehat{s}_2^+(\zeta) (I - \omega(\zeta) \widehat{s}^+(\zeta))^{-1} \omega(\zeta) \widehat{s}_1^+(\zeta) i_{\mathcal{K}'_+} \quad (7.9)$$

where

$$\widehat{s}_0^+(\zeta) = i_{\mathcal{E}''}^* \widehat{S}_0^+(\zeta) i_{\mathcal{G}'}, \quad \widehat{s}_2(\zeta) = i_{\mathcal{E}''}^* \widehat{S}_2^+(\zeta), \quad \widehat{s}_1(\zeta) = \widehat{S}_1^+(\zeta) i_{\mathcal{G}'}, \quad \widehat{s}(\zeta) = \widehat{S}^+(\zeta)$$

and where

$$\begin{bmatrix} \widehat{S}_0^+(\zeta) & \widehat{S}_2^+(\zeta) \\ \widehat{S}_1^+(\zeta) & \widehat{S}^+(\zeta) \end{bmatrix} = \begin{bmatrix} (I - \zeta A_0)^{-1} & \zeta (I - \zeta A_0)^{-1} B_0 \\ C_0 (I - \zeta A_0)^{-1} & \zeta C_0 (I - \zeta A)^{-1} B_0 \end{bmatrix}$$

is the frequency-domain version of the augmented input-output map associated with the unitary colligation  $U_0$  (and hence is completely determined from the problem data) and where

$$\omega(\zeta) = D_1 + \zeta C_1 (I - \zeta A_1)^{-1} B_1$$

is the characteristic function of the free-parameter unitary colligation  $U_1$ . Let us use the notation  $D_X$  for the defect operator  $D_X := (I - X^* X)^{1/2}$  of  $X$ . Further analysis shows that  $\widehat{Y}_+(\zeta)$  has a factorization  $\widehat{Y}_+(\zeta) = Y_{0+}(\zeta) D_X$  where the operator  $\Gamma : D_X k'_+ \mapsto Y_{0+}(\zeta) D_X k'_+$  defines a contraction operator from  $\mathcal{D}_X := \overline{\text{Ran}} D_X$

(viewed as a space of constant functions) into  $H_{\mathcal{E}''}^2$ . Then we have the following form for the parametrization of the lifts:

$$Y: k'_+ \mapsto \begin{bmatrix} Xk'_+ \\ Y_{0+}(\zeta)D_X k'_+ \end{bmatrix} \text{ where } \widehat{Y}_+(\zeta) = Y_{0+}(\zeta)D_X \text{ is given by (7.9).} \quad (7.10)$$

In Sections 6 and 7 of Chapter XIV in [21] or Theorem VI.5.1 in [22], there are derived formulas for a Redheffer coefficient matrix

$$\Psi(\zeta) = \begin{bmatrix} \Psi_{11}(\zeta) & \Psi_{12}(\zeta) \\ \Psi_{21}(\zeta) & \Psi_{22}(\zeta) \end{bmatrix} \quad (7.11)$$

so that the function  $Y_{0+}(\zeta)$  is expressed by the formula

$$Y_{0+}(\zeta) = \Psi_{11}(\zeta) + \Psi_{12}(\zeta)(I - \omega(\zeta)\Psi_{22}(\zeta))^{-1}\omega(\zeta)\Psi_{21}(\zeta). \quad (7.12)$$

S. ter Horst (private communication) has verified that, after some changes of variable, the formula (7.12) agrees with (7.10).

In this formulation of the Lifting Problem, the intertwining property (1.3) is encoded directly in terms of  $\widehat{Y}_{0+}(\zeta)$  in the form

$$Y_{0+}(\zeta)D_X \mathcal{U}_+ = i_{\mathcal{E}''}^* X + \zeta Y_{0+}(\zeta)D_X. \quad (7.13)$$

Here the range of  $i_{\mathcal{E}''}^*$  is the space  $\mathcal{E}''$  and  $\mathcal{E}''$  is identified as the subspace of constant functions in  $H_{\mathcal{E}''}^2$ . Associated with the data of a Lifting Problem is an underlying isometry  $\rho: \mathcal{F} \rightarrow \mathcal{E}'' \oplus \mathcal{D}_X$  where

$$\mathcal{F} = \overline{\text{Ran}} D_X \mathcal{U}'_+$$

and defined densely by

$$\rho D_X \mathcal{U}'_+ k'_+ = \begin{bmatrix} \rho_1 D_X \mathcal{U}'_+ k'_+ \\ \rho_2 D_X \mathcal{U}'_+ k'_+ \end{bmatrix} := \begin{bmatrix} i_{\mathcal{E}''}^* X k'_+ \\ D_X k'_+ \end{bmatrix}. \quad (7.14)$$

Then the form (7.13) of the intertwining condition can be expressed directly in terms of the isometry  $\rho$  in the form

$$\rho_1 + \zeta \cdot Y_{0+}(\zeta)\rho_2 = Y_{0+}(\zeta)|_{\mathcal{F}}. \quad (7.15)$$

It is this formulation which has been extended to the context of the Relaxed Comutant Lifting problem in [25, 26] and in addition to a Redheffer parametrization for the set of all solutions in [29, 30]. For the relaxed problem, the underlying isometry  $\rho$  given by (7.14) is in general only a contraction rather than an isometry. The Redheffer coefficient matrix (7.11) is a coisometry from  $\mathcal{D}_X \oplus H_{\mathcal{D}_{\rho^*}}^2$  to  $H_{\mathcal{E}''}^2 \oplus H_{\mathcal{D}_{T'}}^2$  ( $\Psi_{11}$  and  $\Psi_{21}$  are multiplication operators).

## 8. THE UNIVERSAL EXTENSION

Theorem 7.1 obtained a parametrization of all symbols of solutions of the lifting problem (and therefore also of all lifts under the assumption that  $\mathcal{G}'$  and  $\mathcal{G}''$  are \*-cyclic) via a Redheffer linear-fractional map acting on a free-parameter input-output map, or equivalently, a free-parameter Schur-class function, acting between coefficient spaces  $\widetilde{\Delta}$  and  $\widetilde{\Delta}_*$ . As has been observed before in a variety of contexts (see e.g. [21, 22]), a special role is played by the lift associated with the free-parameter taken to be equal to 0 (the *central lift*). In this section we develop the special properties of the universal lift from the point of view of the ideas developed here.

The first step is to construct the simple unitary colligation having characteristic function equal to the zero function.

**Theorem 8.1.** *The essentially unique simple unitary colligation*

$$U_{10} = \begin{bmatrix} A_{10} & B_{10} \\ C_{10} & D_{10} \end{bmatrix} : \begin{bmatrix} \mathcal{H}_{10} \\ \tilde{\Delta} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_{10} \\ \tilde{\Delta}_* \end{bmatrix} \quad (8.1)$$

having characteristic function equal to the zero function

$$\omega_{10}(\lambda) = D_0 + \lambda C_0(I - \lambda A_0)^{-1} B_0 \equiv 0: \tilde{\Delta} \rightarrow \tilde{\Delta}_*$$

is constructed as follows: take

$$\begin{aligned} \mathcal{H}_{10} &= \begin{bmatrix} \ell_{\tilde{\Delta}_*}^2(\mathbb{Z}_-) \\ \ell_{\tilde{\Delta}}^2(\mathbb{Z}_+) \end{bmatrix}, \quad A_{10} = \begin{bmatrix} J_-^* & 0 \\ 0 & J_+^* \end{bmatrix}, \\ B_{10} &= \begin{bmatrix} i_{\tilde{\Delta}}^{(-1)} \\ 0 \end{bmatrix}, \quad C_{10} = \begin{bmatrix} 0 & i_{\tilde{\Delta}_*}^{(0)*} \end{bmatrix}, \quad D_{10} = 0, \end{aligned} \quad (8.2)$$

where in general  $J_-: (\dots, x(-2), x(-1)) \mapsto (\dots, x(-3), x(-2))$  is the compressed forward shift on  $\ell_{\mathcal{X}}^2(\mathbb{Z}_-)$ ,  $J_+: (x(0), x(1), x(2), \dots) \mapsto (0, x(0), x(1), \dots)$  is the forward shift on  $\ell_{\mathcal{X}}^2(\mathbb{Z}_+)$  (with coefficient space  $\mathcal{X}$  clear from the context), where  $i_{\tilde{\Delta}}^{(-1)}: x \mapsto (\dots, 0, x)$  is the natural injection of  $\tilde{\Delta}$  into the subspace of elements of  $\ell_{\tilde{\Delta}}^2(\mathbb{Z}_-)$  supported on the singleton  $\{-1\}$ , and  $i_{\tilde{\Delta}_*}^{(0)*}: x \mapsto (x, 0, 0, \dots)$  is the natural injection of  $\tilde{\Delta}_*$  into the subspace of elements of  $\ell_{\tilde{\Delta}_*}^2(\mathbb{Z}_+)$  supported on the singleton  $\{0\}$ .

*Proof.* This is a straightforward verification which we leave to the reader.  $\square$

We have seen in Theorem 6.1 that the operator  $\mathcal{U}^*$  on  $\mathcal{K}$  extends the isometry  $V$  on  $\mathcal{H}_0$  having domain  $\mathcal{D} \subset \mathcal{H}_0$  and range  $\mathcal{D}_* \subset \mathcal{H}_0$  if and only if  $\mathcal{U}^*$  has a representation of the form

$$\mathcal{U}^* = \mathcal{F}_\ell(U_0, U_1)$$

where  $U_0$  is the universal colligation given by (6.4) or equivalently by (6.5) and (6.6) and where  $U_1: \begin{bmatrix} \mathcal{H}_1 \\ \tilde{\Delta} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \tilde{\Delta}_* \end{bmatrix}$  is a free-parameter unitary colligation, and, moreover,  $\mathcal{U}^*$  is a minimal unitary extension of  $V$  if and only if  $U_1$  is a simple unitary colligation. We now consider the particular case where we take  $U_1$  equal to the simple unitary colligation with zero characteristic function  $U_{10}$  given as in Theorem 8.1. We refer to the resulting minimal unitary extension  $\mathcal{U}_0^* := \mathcal{F}_\ell(U_0, U_{10})$  as the *central unitary extension*. An application of the general formula (6.7) then gives

$$\begin{aligned} \mathcal{U}_0^* = \mathcal{F}_\ell(U_0, U_{10}) &= \begin{bmatrix} A_0 + B_0 D_{10} C_0 & B_0 C_{10} \\ B_{10} C_0 & A_{10} \end{bmatrix} \\ &= \begin{bmatrix} A_0 & 0 & i_{\tilde{\Delta}_*} i_{\tilde{\Delta}}^{(0)*} \\ i_{\tilde{\Delta}}^{(-1)} i_{\tilde{\Delta}}^* & J_-^* & 0 \\ 0 & 0 & J_+^* \end{bmatrix} \text{ on } \mathcal{K}_0 := \begin{bmatrix} \mathcal{H}_0 \\ \ell_{\tilde{\Delta}}^2(\mathbb{Z}_-) \\ \ell_{\tilde{\Delta}_*}^2(\mathbb{Z}_+) \end{bmatrix}. \end{aligned} \quad (8.3)$$



with adjoint given by

$$\mathcal{U}_0 = \begin{bmatrix} A_0^* & i_{\tilde{\Delta}} i_{\tilde{\Delta}}^{(-1)*} & 0 \\ 0 & J_- & 0 \\ i_{\tilde{\Delta}_*}^{(0)} i_{\tilde{\Delta}_*}^* & 0 & J_+ \end{bmatrix}. \quad (8.4)$$

To analyze the finer structure of the universal extension  $(\mathcal{U}_0, \mathcal{K}_0)$  given by (8.4), let us define embedding operators

$$i_{\tilde{\Delta},0}: \tilde{\Delta} \rightarrow \mathcal{K}_0, \quad i_{\tilde{\Delta}_*,0}: \tilde{\Delta}_* \rightarrow \mathcal{K}_0, \quad i_{\mathcal{K}''_-,0}: \mathcal{K}''_- \rightarrow \mathcal{K}_0, \quad i_{\mathcal{K}'_+,0}: \mathcal{K}'_+ \rightarrow \mathcal{K}_0$$

by

$$i_{\tilde{\Delta},0} = \begin{bmatrix} 0 \\ i_{\tilde{\Delta}}^{(-1)} \\ 0 \end{bmatrix}, \quad i_{\tilde{\Delta}_*,0} = \begin{bmatrix} 0 \\ 0 \\ i_{\tilde{\Delta}_*}^{(0)} \end{bmatrix}, \quad i_{\mathcal{K}''_-,0} = \begin{bmatrix} i_{\mathcal{K}''_- \rightarrow \mathcal{H}_0} \\ 0 \\ 0 \end{bmatrix}, \quad i_{\mathcal{K}'_+,0} = \begin{bmatrix} i_{\mathcal{K}'_+ \rightarrow \mathcal{H}_0} \\ 0 \\ 0 \end{bmatrix}.$$

Then the collection

$$\mathfrak{S}_0 = (\mathcal{U}_0, [i_{\tilde{\Delta},0} \quad i_{\tilde{\Delta}_*,0} \quad i_{\mathcal{K}''_-,0} \quad i_{\mathcal{K}'_+,0}]; \mathcal{K}_0, \tilde{\Delta} \oplus \tilde{\Delta}_* \oplus \mathcal{K}''_- \oplus \mathcal{K}'_+) \quad (8.5)$$

is a scattering system in the sense of Section 3.1 (see (3.1)). Moreover, the operators  $i_{\tilde{\Delta},0}, i_{\tilde{\Delta}_*,0}, i_{\mathcal{K}''_-,0}, i_{\mathcal{K}'_+,0}$  have unique respective isometric extensions

$$\vec{i}_{\tilde{\Delta},0}: \ell_{\tilde{\Delta}}^2(\mathbb{Z}) \rightarrow \mathcal{K}_0, \quad \vec{i}_{\tilde{\Delta}_*,0}: \ell_{\tilde{\Delta}_*}^2(\mathbb{Z}) \rightarrow \mathcal{K}_0, \quad i_{\mathcal{K}''_-,0}: \mathcal{K}''_- \rightarrow \mathcal{K}_0, \quad i_{\mathcal{K}'_+,0}: \mathcal{K}'_+ \rightarrow \mathcal{K}_0$$

which satisfy the respective intertwining conditions

$$\vec{i}_{\tilde{\Delta},0} J = \mathcal{U}_0 \vec{i}_{\tilde{\Delta},0}, \quad \vec{i}_{\tilde{\Delta}_*,0} J = \mathcal{U}_0 \vec{i}_{\tilde{\Delta}_*,0}, \quad i_{\mathcal{K}''_-,0} \mathcal{U}'' = \mathcal{U}_0 i_{\mathcal{K}''_-,0}, \quad i_{\mathcal{K}'_+,0} \mathcal{U}' = \mathcal{U}_0 i_{\mathcal{K}'_+,0}$$

where here we set  $J$  equal to the bilateral shift operator on any space of the form  $\ell_{\mathcal{X}}^2(\mathbb{Z})$  (the coefficient space  $\mathcal{X}$  determined by the context). Then the collection

$$\mathfrak{S}_{AA,0} = (\mathcal{U}_0, \vec{i}_{\tilde{\Delta},0}, \vec{i}_{\tilde{\Delta}_*,0}, i_{\mathcal{K}''_-,0}, i_{\mathcal{K}'_+,0}; \mathcal{K}_0) \quad (8.6)$$

can be viewed as a *four-fold AA-unitary coupling* of the four unitary operators

$$(J, \ell_{\tilde{\Delta}}^2(\mathbb{Z})), \quad (J, \ell_{\tilde{\Delta}_*}^2(\mathbb{Z})), \quad (\mathcal{U}'', \mathcal{K}''), \quad (\mathcal{U}', \mathcal{K}')$$

which has certain additional properties. The next theorem identifies some of these additional properties.

**Theorem 8.2.** *The scattering system (8.5) and its extension to the four-fold AA-unitary coupling (8.6) associated with the universal extension (8.4)  $\mathcal{U}_0$  for a Lifting Problem have the following properties:*

(1) *The density conditions*

$$\text{im } i_{\mathcal{K}'_+,0} + \text{im } i_{\mathcal{K}''_-,0} \text{ is dense in } \mathcal{K}_0, \quad (8.7)$$

$$\begin{aligned} \overline{\text{span}}\{\text{im } i_{\mathcal{K}'_+,0}, \text{im } i_{\mathcal{K}''_-,0}\} &= \mathcal{K}_0 \ominus \vec{i}_{\tilde{\Delta}_*,0}(\ell_{\tilde{\Delta}_*}^2(\mathbb{Z}_+)), \\ \overline{\text{span}}\{\text{im } i_{\mathcal{K}'_+,0}, \text{im } i_{\mathcal{K}''_-,0}\} &= \mathcal{K}_0 \ominus \vec{i}_{\tilde{\Delta},0}(\ell_{\tilde{\Delta}}^2(\mathbb{Z}_-)), \end{aligned} \quad (8.8)$$

and

$$\overline{\text{span}}\{\text{im } i_{\mathcal{K}'_+,0}, \text{im } i_{\mathcal{K}''_-,0}\} \cap \overline{\text{span}}\{\text{im } i_{\mathcal{K}'_+,0}, \text{im } i_{\mathcal{K}''_-,0}\} = \overline{\text{span}}\{\text{im } i_{\mathcal{K}'_+,0}, \text{im } i_{\mathcal{K}''_-,0}\} \quad (8.9)$$

hold.

(2) *The orthogonality conditions*

$$\begin{aligned} \vec{i}_{\tilde{\Delta}_*,0}(\ell_{\tilde{\Delta}_*}^2(\mathbb{Z}_+)) \perp \text{im } i_{\mathcal{K}''_-,0}, \quad \vec{i}_{\tilde{\Delta},0}(\ell_{\tilde{\Delta}}^2(\mathbb{Z}_-)) \perp \text{im } i_{\mathcal{K}'_+,0}, \\ \vec{i}_{\tilde{\Delta},0}(\ell_{\tilde{\Delta}}^2(\mathbb{Z}_-)) \perp \vec{i}_{\tilde{\Delta}_*,0}(\ell_{\tilde{\Delta}_*}^2(\mathbb{Z}_+)) \end{aligned} \quad (8.10)$$

and

$$\vec{i}_{\tilde{\Delta},0}(\ell_{\tilde{\Delta}}^2(\mathbb{Z}_-)) \perp \text{im } i_{\mathcal{K}''_-,0}, \quad \vec{i}_{\tilde{\Delta}_*,0}(\ell_{\tilde{\Delta}_*}^2(\mathbb{Z}_+)) \perp \text{im } i_{\mathcal{K}'_+,0} \quad (8.11)$$

hold.

(3) *The subspace identities*

$$\mathcal{U}_0^* \text{im } i_{\tilde{\Delta}_*,0} = i_{\mathcal{H}_0,0} \Delta_*, \quad \mathcal{U}_0 \text{im } i_{\tilde{\Delta},0} = i_{\mathcal{H}_0,0} \Delta \quad (8.12)$$

hold

*Proof.* For simplicity let us use the bold notation

$$\mathbf{H}_0 = \text{im } i_{\mathcal{H}_0,0}$$

to indicate the subspace  $\mathcal{H}_0$  when viewed as a subspace of  $\mathcal{K}_0$ .

Property (8.7) is a consequence of the fact that the universal free-parameter unitary colligation  $U_{10}$  given by (8.1) and (8.2) is simple and hence (by Theorem 6.1) the unitary operator  $\mathcal{U}^* = \mathcal{F}_\ell(U_0, U_{10})$  is a minimal unitary extension of  $V$ .

To check conditions (8.8), we use the orthogonal decomposition of  $\mathcal{K}_0$  (see (8.3))

$$\mathcal{K}_0 = \vec{i}_{\tilde{\Delta},0}(\ell_{\tilde{\Delta}}^2(\mathbb{Z}_-)) \oplus \text{im } i_{\mathcal{H}_0,0} \oplus \vec{i}_{\tilde{\Delta}_*,0}(\ell_{\tilde{\Delta}_*}^2(\mathbb{Z}_+)). \quad (8.13)$$

From the formula for  $\mathcal{U}_0^*$  in (8.3), it is easily checked that the smallest  $\mathcal{U}_0$ -invariant subspace  $\mathbf{H}_{0+}$  containing  $\mathbf{H}_0$  is

$$\mathbf{H}_{0+} = \mathbf{H}_0 \oplus \vec{i}_{\tilde{\Delta}_*,0}(\ell_{\tilde{\Delta}_*}^2(\mathbb{Z}_+)) = \mathcal{K}_0 \ominus \text{im } \vec{i}_{\tilde{\Delta},0}(\ell_{\tilde{\Delta}}^2(\mathbb{Z}_-)).$$

On the other hand, by the construction this smallest  $\mathcal{U}_0$ -invariant subspace can also be identified as  $\mathbf{H}_{0+} = \overline{\text{span}}\{\text{im } i_{\mathcal{K}''_-,0}, \text{im } i_{\mathcal{K}'_+,0}\}$ . Combining these observations gives the first part of (8.8). The second part follows similarly by identifying the smallest  $\mathcal{U}_0^*$ -invariant subspace of  $\mathbf{H}_{0-}$  containing  $\mathbf{H}_0$  as  $\mathbf{H}_{0-} = \mathcal{K}_0 \ominus \vec{i}_{\tilde{\Delta}_*,0}(\ell_{\tilde{\Delta}_*}^2(\mathbb{Z}_+))$  on the one hand and also as  $\mathbf{H}_{0-} = \overline{\text{span}}\{\text{im } i_{\mathcal{K}''_-,0}, \text{im } i_{\mathcal{K}'_+,0}\}$  on the other. To prove (8.9), note from the above discussion that

$$\mathbf{H}_{0+} = \mathbf{H}_0 \oplus \vec{i}_{\tilde{\Delta}_*,0}(\ell_{\tilde{\Delta}_*}^2(\mathbb{Z}_+)),$$

$$\mathbf{H}_{0-} = \mathbf{H}_0 \oplus \vec{i}_{\tilde{\Delta},0}(\ell_{\tilde{\Delta}}^2(\mathbb{Z}_-)).$$

As  $\vec{i}_{\tilde{\Delta}_*,0}(\ell_{\tilde{\Delta}_*}^2(\mathbb{Z}_+))$  and  $\vec{i}_{\tilde{\Delta},0}(\ell_{\tilde{\Delta}}^2(\mathbb{Z}_-))$  are orthogonal to each other, it follows that  $\mathbf{H}_{0+} \cap \mathbf{H}_{0-} = \mathbf{H}_0$ , i.e., (8.9) holds.

The orthogonality conditions (8.10) and (8.11) are clear from (8.13). In fact, the orthogonality conditions (8.11) hold in the stronger form

$$\text{im } \vec{i}_{\tilde{\Delta}_*,0} \perp \text{im } i_{\mathcal{K}'_+,0}, \quad \text{im } \vec{i}_{\tilde{\Delta},0} \perp \text{im } i_{\mathcal{K}''_-,0}. \quad (8.14)$$

To see this, note that  $\mathcal{K}'_+$  is invariant under  $\mathcal{U}'$  and  $i_{\mathcal{K}'_+,0} \mathcal{U}' = \mathcal{U}_0 i_{\mathcal{K}'_+,0}$  and hence  $\text{im } i_{\mathcal{K}'_+,0}$  is invariant under  $\mathcal{U}_0$  and the first of the orthogonality conditions (8.11) implies that  $\mathcal{U}_0^{*n} \vec{i}_{\tilde{\Delta}_*,0}(\ell_{\tilde{\Delta}_*}^2(\mathbb{Z}_+))$  is orthogonal to  $\text{im } i_{\mathcal{K}'_+,0}$ . As the subspace

$$\cup_{n=0}^{\infty} \mathcal{U}_0^{*n} \vec{i}_{\tilde{\Delta}_*,0}(\ell_{\tilde{\Delta}_*}^2(\mathbb{Z}_+)) = \cup_{n=0}^{\infty} \vec{i}_{\tilde{\Delta},0}(J^{*n} \ell_{\tilde{\Delta}_*}^2(\mathbb{Z}_+))$$

is dense in  $\text{im } i_{\tilde{\Delta}_*,0}^{\vec{}}$ , we conclude that  $\text{im } i_{\tilde{\Delta}_*,0}^{\vec{}}$  is orthogonal to  $\text{im } i_{\mathcal{K}'_+,0}$ . As  $\text{im } i_{\tilde{\Delta}_*,0}^{\vec{}}$  is reducing for  $\mathcal{U}_0$ , we conclude that in fact  $\text{im } i_{\tilde{\Delta}_*,0}^{\vec{}}$  is orthogonal to the smallest  $\mathcal{U}_0$ -reducing subspace containing  $\text{im } i_{\mathcal{K}'_+,0}$ , i.e., to  $\text{im } i_{\mathcal{K}',0}$ , and the first of conditions (8.14) follows. The second orthogonality condition in (8.14) follows similarly from the observation that  $\text{im } i_{\mathcal{K}''_-}$  is invariant under  $\mathcal{U}''^*$ .

The subspace identities (8.12) can be read off from the definitions, in particular, the definition of  $\mathcal{U}_0$  (6.4).  $\square$

**Remark 8.3.** One can easily verify that the orthogonality conditions (8.10) and (8.11) can be expressed in more succinct fashion as

$$i_{\mathcal{K}''_-,0}^* \mathcal{U}_0^{*n} i_{\tilde{\Delta}_*,0} = 0 \text{ for } n \leq 0, \quad (8.15)$$

$$i_{\tilde{\Delta}_-,0}^* \mathcal{U}_0^{*n} i_{\mathcal{K}'_+,0} = 0 \text{ for } n < 0, \quad (8.16)$$

$$i_{\tilde{\Delta}_-,0}^* \mathcal{U}_0^{*n} i_{\tilde{\Delta}_*,0} = 0 \text{ for } n < 0, \quad (8.17)$$

and

$$i_{\tilde{\Delta}_-,0}^* \mathcal{U}_0^{*n} i_{\mathcal{K}''_-,0} = 0 \text{ for } n \geq 0, \quad (8.18)$$

$$i_{\tilde{\Delta}_*,0}^* \mathcal{U}_0^{*n} i_{\mathcal{K}'_+,0} = 0 \text{ for } n < 0. \quad (8.19)$$

Since actually the stronger relations (8.14) hold, the conditions (8.18) and (8.19) actually hold for all  $n \in \mathbb{Z}$ :

$$i_{\tilde{\Delta}_-,0}^* \mathcal{U}_0^{*n} i_{\mathcal{K}''_-,0} = 0 \text{ for all } n \in \mathbb{Z}, \quad (8.20)$$

$$i_{\tilde{\Delta}_*,0}^* \mathcal{U}_0^{*n} i_{\mathcal{K}'_+,0} = 0 \text{ for all } n \in \mathbb{Z}, \quad (8.21)$$

respectively.

It is of interest that conversely the properties (8.7), (8.8), (8.10), (8.11) and (8.12) can be used to characterize the universal extension  $\mathcal{U}_0$  associated with a Lifting Problem. We present two versions of such a result.

**Theorem 8.4.** *Suppose that  $(\mathcal{U}'', \mathcal{K}'')$  and  $(\mathcal{U}', \mathcal{K}')$  are unitary operators and that  $\mathcal{K}''_- \subset \mathcal{K}''$  and  $\mathcal{K}'_+ \subset \mathcal{K}'$  are  $*$ -cyclic subspaces with  $\mathcal{K}''_-$  and  $\mathcal{K}'_+$  invariant under  $\mathcal{U}''^*$  and  $\mathcal{U}'$  respectively. Suppose also that  $\tilde{\Delta}$  and  $\tilde{\Delta}_*$  are two coefficient Hilbert spaces and that we are given a scattering system of the form*

$$\mathfrak{S}_0 = (\mathcal{U}_0, \begin{bmatrix} i_{\tilde{\Delta},0} & i_{\tilde{\Delta}_*,0} & i_{\mathcal{K}''_-,0} & i_{\mathcal{K}'_+,0} \end{bmatrix}; \mathcal{K}_0, \tilde{\Delta} \oplus \tilde{\Delta}_* \oplus \mathcal{K}''_- \oplus \mathcal{K}'_+)$$

where  $i_{\tilde{\Delta},0}$ ,  $i_{\tilde{\Delta}_*,0}$ ,  $i_{\mathcal{K}''_-,0}$  and  $i_{\mathcal{K}'_+,0}$  are isometric embedding operators of the respective spaces  $\tilde{\Delta}$ ,  $\tilde{\Delta}_*$ ,  $\mathcal{K}''_-$  and  $\mathcal{K}'_+$  into  $\mathcal{K}_0$ . We assume also that there is a four-fold AA-unitary coupling

$$\mathfrak{S}_{AA,0} = \left( \mathcal{U}_0, \begin{bmatrix} i_{\tilde{\Delta},0}^{\vec{}} & i_{\tilde{\Delta}_*,0}^{\vec{}} & i_{\mathcal{K}''_-,0} & i_{\mathcal{K}'_+,0} \end{bmatrix}; \mathcal{K}_0 \right)$$

of the four unitary operators

$$(J, \ell_{\tilde{\Delta}}^2(\mathbb{Z})), \quad (J, \ell_{\tilde{\Delta}_*}^2(\mathbb{Z})), \quad (\mathcal{U}'', \mathcal{K}''), \quad (\mathcal{U}', \mathcal{K}')$$

which extends  $\mathfrak{S}_0$  in the sense that

$$i_{\tilde{\Delta},0} = i_{\tilde{\Delta},0}^{\vec{}} \circ i_{\tilde{\Delta}}^{(-1)}, \quad i_{\tilde{\Delta}_*,0} = i_{\tilde{\Delta}_*,0}^{\vec{}} \circ i_{\tilde{\Delta}_*}^{(0)}, \quad i_{\mathcal{K}''_-,0}|_{\mathcal{K}''_-} = i_{\mathcal{K}''_-,0}, \quad i_{\mathcal{K}'_+,0}|_{\mathcal{K}'_+} = i_{\mathcal{K}'_+,0}. \quad (8.22)$$

Define subspaces  $\mathcal{H}_0$ ,  $\mathcal{D}$ ,  $\mathcal{D}_*$ ,  $\Delta$  and  $\Delta_*$  of  $\mathcal{K}_0$  according to

$$\begin{aligned}\mathcal{H}_0 &= \text{clos im } [i_{\mathcal{K}'_-,0} \ i_{\mathcal{K}'_+,0}], \quad \mathcal{D} = \text{clos} \left( \text{im } i_{\mathcal{K}''_-,0} + i_{\mathcal{K}'_+,0}(\mathcal{U}'\mathcal{K}'_+) \right), \\ \mathcal{D}_* &= \text{clos} \left( i_{\mathcal{K}''_-,0}(\mathcal{U}''^*\mathcal{K}''_-) + \text{im } i_{\mathcal{K}'_+,0} \right), \quad \Delta = \mathcal{H}_0 \ominus \mathcal{D}, \quad \Delta_* = \mathcal{H}_0 \ominus \mathcal{D}_*\end{aligned}\quad (8.23)$$

and let  $i_{\mathcal{H}_0,0}: \mathcal{H}_0 \rightarrow \mathcal{K}_0$  be the isometric inclusion map. Suppose also that either one of the the following additional conditions holds:

- (1) Conditions (8.7), (8.10), (8.11), and (8.12) all hold, or
- (2) Conditions (8.8), (8.9), (8.10), (8.11), and the following weaker form of (8.12)

$$\text{im } i_{\tilde{\Delta}_*,0} \perp \text{im } \mathcal{U}i_{\tilde{\Delta},0} \quad (8.24)$$

hold.

Then  $\mathfrak{S}_0$  and  $\mathfrak{S}_{AA,0}$  are equal to the scattering system and the four-fold AA-unitary coupling associated with the universal extension  $\mathcal{U}_0^*$  from some Lifting Problem.

**Remark 8.5.** From the first version of Theorem 8.4, we see that if (8.7), (8.10), (8.11) and (8.12) hold, then also (8.14), (8.8) and (8.9) hold. From the second version, we see that if (8.8), (8.9), (8.10), (8.11) and (8.24) hold, then also (8.14), (8.7) and (8.12) hold.

In the proofs below it is convenient to use the bold notation

$$\mathbf{H}_0 = \text{im } i_{\mathcal{H}_0}, \quad \mathbf{D} = i_{\mathcal{H}_0,0}(\mathcal{D}), \quad \mathbf{D}_* = i_{\mathcal{H}_0,0}(\mathcal{D}_*), \quad \mathbf{\Delta} = i_{\mathcal{H}_0}(\Delta), \quad \mathbf{\Delta}_* = i_{\mathcal{H}_0,0}(\Delta_*)$$

for the subspaces introduced in (8.23) when viewed as subspaces of  $\mathcal{K}_0$  rather than of  $\mathcal{H}_0$ , as well as the additional simplifications

$$\mathcal{G}_- = \vec{i}_{\tilde{\Delta},0}(\ell_{\tilde{\Delta}}^2(\mathbb{Z}_-)) \subset \mathcal{K}_0, \quad \mathcal{G}_{*+} = \vec{i}_{\tilde{\Delta}_*,0}(\ell_{\tilde{\Delta}_*}^2(\mathbb{Z}_+)) \subset \mathcal{K}_0.$$

*Proof of version 1:* The combined effect of the hypotheses (8.10) and (8.11) is that the three subspaces  $\mathbf{H}_0, \mathcal{G}_-, \mathcal{G}_{*+}$  are pairwise orthogonal. Therefore the span of these subspaces  $\mathcal{K}_{00}$  has an orthogonal decomposition

$$\mathcal{K}_{00} = \mathbf{H}_0 \oplus \mathcal{G}_- \oplus \mathcal{G}_{*+}. \quad (8.25)$$

From the definitions we see that  $\mathbf{H}_0$  has a two-fold orthogonal decomposition as

$$\mathbf{H}_0 = \mathbf{D} \oplus \mathbf{\Delta} = \mathbf{D}_* \oplus \mathbf{\Delta}_*. \quad (8.26)$$

Due to the intertwining

$$\mathcal{U}_0 i_{\mathcal{K}''_-,0} = i_{\mathcal{K}''_-,0} \mathcal{U}''', \quad \mathcal{U}_0 i_{\mathcal{K}'_+,0} = i_{\mathcal{K}'_+,0} \mathcal{U}'$$

one can see that

$$\mathcal{U}_0^*(\mathbf{D}) = \mathbf{D}_* \quad (8.27)$$

and in fact we have the alternate characterizations of  $\mathbf{D}$  and  $\mathbf{D}_*$ :

$$\mathbf{D} = \{h \in \mathbf{H}_0: \mathcal{U}_0^* h \in \mathbf{H}_0\}, \quad \mathbf{D}_* = \{h_* \in \mathbf{H}_0: \mathcal{U}_0 h_* \in \mathbf{H}_0\}. \quad (8.28)$$

From the hypothesis (8.12) we know that

$$\mathcal{U}_0^* \left( \text{im } i_{\tilde{\Delta}_*,0} \right) = \mathbf{\Delta}_*, \quad \mathcal{U}_0^* \mathbf{\Delta} = \text{im } i_{\tilde{\Delta},0} \quad (8.29)$$

and, from the intertwining  $\mathcal{U}_0^* \vec{i}_{\tilde{\Delta}_*,0} = \vec{i}_{\tilde{\Delta}_*,0} J^*$  and  $\mathcal{U}_0^* \vec{i}_{\tilde{\Delta},0} = \vec{i}_{\tilde{\Delta},0} J^*$ , we know that

$$\mathcal{G}_- = \mathcal{U}_0^* \mathcal{G}_- \oplus \text{im } \vec{i}_{\tilde{\Delta},0}, \quad \mathcal{G}_{*+} = \text{im } \vec{i}_{\tilde{\Delta}_*,0} \oplus \mathcal{U}_0 \mathcal{G}_{*+}. \quad (8.30)$$

From the orthogonal decompositions (8.25) and (8.26) for  $\mathcal{K}_0$  and  $\mathcal{H}_0$  combined with (8.27), (8.29) and (8.30) we see that  $\mathcal{K}_{00}$  is reducing for  $\mathcal{U}_0$ . From hypothesis (8.7) we conclude that in fact  $\mathcal{K}_{00} = \mathcal{K}_0$  and the decomposition (8.25) applies with  $\mathcal{K}_0$  in place of  $\mathcal{K}_{00}$ , i.e., we have

$$\mathcal{K}_0 = \mathcal{G}_- \oplus \mathcal{H}_0 \oplus \mathcal{G}_{*+}. \quad (8.31)$$

From (8.27) we see that we may define an isometry  $V$  on  $\mathcal{H}_0$  with domain  $\mathcal{D}$  and range  $\mathcal{D}_*$  by

$$Vd = d_* \text{ if } \mathcal{U}_0^* i_{\mathcal{H}_0,0} d = i_{\mathcal{H}_0,0} d_* \text{ for } d \in \mathcal{D}, d_* \in \mathcal{D}_*.$$

It is now straightforward to check that necessarily  $\mathcal{U}_0^*$  is a universal extension of the isometry  $V$ . Furthermore, one can check that  $V$  is the isometry constructed from the Lifting Problem data

$$X = i_{\mathcal{K}_-'' ,0}^* i_{\mathcal{K}_+' ,0}, \quad (\mathcal{U}'', \mathcal{K}''), \quad (\mathcal{U}', \mathcal{K}'), \quad \mathcal{K}_+' \subset \mathcal{K}', \quad \mathcal{K}_-'' \subset \mathcal{K}''.$$

This completes the proof of the first version of Theorem 8.4.  $\square$

*Proof of version 2:* Using hypotheses (8.10) and (8.11) as in the proof of version 1, we form the subspace  $\mathcal{K}_{00}$  as in (8.25). If we define  $\tilde{\mathcal{H}}_0$  by

$$\tilde{\mathcal{H}}_0 := \mathcal{K}_0 \ominus [\mathcal{G}_- \oplus \mathcal{G}_{*+}],$$

then by definition we have

$$\mathcal{K}_0 = \tilde{\mathcal{H}}_0 \oplus \mathcal{G}_- \oplus \mathcal{G}_{*+}. \quad (8.32)$$

For convenience let us introduce the temporary notation

$$\mathcal{H}_{0-} = \overline{\text{span}}\{\text{im } i_{\mathcal{K}' ,0}, \text{im } i_{\mathcal{K}_-'' ,0}\}, \quad (8.33)$$

$$\mathcal{H}_{0+} = \overline{\text{span}}\{\text{im } i_{\mathcal{K}_+' ,0}, \text{im } i_{\mathcal{K}'' ,0}\}. \quad (8.34)$$

Note that  $\mathcal{H}_{0-}$  is the smallest  $\mathcal{U}_0^*$ -invariant subspace of  $\mathcal{K}_0$  containing  $\mathcal{H}_0$  and that  $\mathcal{H}_{0+}$  is the smallest  $\mathcal{U}_0$ -invariant subspace of  $\mathcal{K}_0$  containing  $\mathcal{H}_0$ . Hypothesis (8.8) now takes the form

$$\mathcal{K}_0 = \mathcal{H}_{0-} \oplus \mathcal{G}_{*+}, \quad (8.35)$$

$$\mathcal{K}_0 = \mathcal{H}_{0+} \oplus \mathcal{G}_-. \quad (8.36)$$

Combining (8.35) and (8.36) with (8.32) gives

$$\mathcal{H}_{0-} = \tilde{\mathcal{H}}_0 \oplus \mathcal{G}_-, \quad \mathcal{H}_{0+} = \tilde{\mathcal{H}}_0 \oplus \mathcal{G}_{*+}.$$

Since  $\mathcal{G}_-$  is orthogonal to  $\mathcal{G}_{*+}$  in  $\mathcal{K}_0$ , we then get

$$\mathcal{H}_{0-} \cap \mathcal{H}_{0+} = \tilde{\mathcal{H}}_0.$$

We may now invoke (8.9) to conclude that  $\mathcal{H}_0 = \tilde{\mathcal{H}}_0$  and hence also  $\mathcal{K}_{00} = \mathcal{K}_0$  and  $\mathcal{K}_0$  has the orthogonal decomposition (8.25) (with  $\mathcal{K}_0$  in place of  $\mathcal{K}_{00}$ ), i.e. (8.31) holds.

From the third condition in (8.10) combined with (8.24), we see that in fact

$$\mathcal{U}_0^* \text{im } i_{\tilde{\Delta}_+,0} \perp \mathcal{G}_-, \quad \mathcal{U} \text{im } i_{\tilde{\Delta},0} \perp \mathcal{G}_{*+}.$$

But also

$$\mathcal{U}^* \text{im } i_{\tilde{\Delta}_+,0} \perp \mathcal{G}_{*+}, \quad \mathcal{U} \text{im } i_{\tilde{\Delta},0} \perp \mathcal{G}_-.$$

Hence we have

$$\mathcal{U}^* \text{im } i_{\tilde{\Delta}_+,0} \perp \mathcal{G}_- \oplus \mathcal{G}_{*+}, \quad \mathcal{U} \text{im } i_{\tilde{\Delta},0} \perp \mathcal{G}_- \oplus \mathcal{G}_{*+}.$$

From the orthogonal decomposition for  $\mathcal{K}_0$  (8.13), we conclude that

$$\mathcal{U}_0^* \text{im } i_{\tilde{\Delta}_*,0} \subset \mathcal{H}_0, \quad \mathcal{U}_0 \text{im } i_{\tilde{\Delta},0} \subset \mathcal{H}_0. \quad (8.37)$$

As in the proof of version (1), we see that  $\mathcal{D}$  and  $\mathcal{D}_*$  have the characterizations (8.28) and  $\mathcal{H}_0$  has the two orthogonal decompositions (8.26). By combining these observations with the decomposition (8.25) for  $\mathcal{K}_0$  and the fact that  $\mathcal{U}_0$  is unitary on  $\mathcal{K}_{00}$ , we see that the containments (8.37) actually force

$$\mathcal{U}_0^* \text{im } i_{\tilde{\Delta}_*,0} = \mathbf{\Delta}_*, \quad \mathcal{U}_0 \text{im } i_{\tilde{\Delta},0} = \mathbf{\Delta},$$

i.e., (8.12) holds. From (8.12) combined with the already proved decomposition (8.31) for  $\mathcal{K}_0$  we see that (8.7) holds as well.

It now follows from the already proved version (1) of Theorem 8.4 that  $(\mathcal{U}_0, \mathcal{K}_0)$  is the central lift with associated central scattering system  $\mathfrak{S}_0$  and four-fold AA-unitary coupling  $\mathfrak{S}_{AA,0}$  coming from a Lifting Problem as asserted.  $\square$

Theorem 7.1 gives a parametrization of the set of all symbols  $w_Y$  (with respect to a choice of two scale subspaces  $\mathcal{G}'' \subset \mathcal{K}_-''$  and  $\mathcal{G}' \subset \mathcal{K}_+'$ ) via a Redheffer-type linear-fractional-transformation (7.4)

$$w_Y = s_0 + s_1(I - \omega s)^{-1} \omega s_2$$

where  $\omega: \ell_{\tilde{\Delta}}(\mathbb{Z}_+) \rightarrow \ell_{\tilde{\Delta}_*}(\mathbb{Z}_+)$  is the input-output map for a free-parameter unitary colligation, and where the Redheffer coefficient matrix (see (7.6))

$$\begin{bmatrix} s_0 & s_2 \\ s_1 & s \end{bmatrix} : \begin{bmatrix} \mathcal{G}' \\ \ell_{\tilde{\Delta}_*}(\mathbb{Z}_+) \end{bmatrix} \rightarrow \begin{bmatrix} \ell_{\mathcal{G}''}(\mathbb{Z}) \\ \ell_{\tilde{\Delta}}(\mathbb{Z}) \end{bmatrix}$$

is completely determined from the Lifting-Problem data. Note that, if elements of the space  $\ell_{\mathcal{G}'' \oplus \tilde{\Delta}}(\mathbb{Z})$  are expressed as infinite column vectors, the first column of the Redheffer coefficient matrix

$$\begin{bmatrix} s_0 \\ s_1 \end{bmatrix} : \mathcal{G}' \rightarrow \begin{bmatrix} \ell_{\mathcal{G}''}(\mathbb{Z}) \\ \ell_{\tilde{\Delta}}(\mathbb{Z}) \end{bmatrix} =: \ell_{\mathcal{G}'' \oplus \tilde{\Delta}}(\mathbb{Z})$$

can be expressed naturally as a column matrix

$$\begin{bmatrix} s_0 \\ s_1 \end{bmatrix} = \text{col}_{n \in \mathbb{Z}} \begin{bmatrix} s_0(n) \\ s_1(n) \end{bmatrix}.$$

If we also view elements of  $\ell_{\tilde{\Delta}_*}(\mathbb{Z}_+)$  as infinite column vectors, then the second column of the Redheffer coefficient matrix

$$\begin{bmatrix} s_2 \\ s \end{bmatrix} : \ell_{\tilde{\Delta}_*}(\mathbb{Z}_+) \rightarrow \ell_{\mathcal{G}'' \oplus \tilde{\Delta}}(\mathbb{Z})$$

can be expressed as an infinite matrix which has Toeplitz structure:

$$\begin{bmatrix} s_2 \\ s \end{bmatrix}_{n,m} = \begin{bmatrix} s_2 \\ s \end{bmatrix}_{n-m,0} =: \begin{bmatrix} s_2(n-m) \\ s(n-m) \end{bmatrix} \text{ for } n \in \mathbb{Z}, m \in \mathbb{Z}_+. \quad (8.38)$$

Let us define the *Redheffer coefficient-matrix symbol* to be simply the operator sequence

$$\left\{ \begin{bmatrix} s_0(n) & s_2(n) \\ s_1(n) & s(n) \end{bmatrix} \right\}_{n \in \mathbb{Z}}. \quad (8.39)$$

The following result shows how the Redheffer coefficient-matrix symbol can be expressed directly in terms of the universal extension  $\mathcal{U}_0$ . To this end we introduce the notation

$$i_{\mathcal{G}',0}: \mathcal{G}' \rightarrow \mathcal{K}_0, \quad i_{\mathcal{G}'',0}: \mathcal{G}'' \rightarrow \mathcal{K}_0$$

for the inclusion of  $\mathcal{G}'$  in  $\mathcal{K}_0$  obtained as the composition  $i_{\mathcal{G}',0} = i_{\mathcal{H}_0,0} i_{\mathcal{G}'}$  of the inclusion of  $\mathcal{G}'$  in  $\mathcal{H}_0$  followed by the inclusion of  $\mathcal{H}_0$  in  $\mathcal{K}_0$ , and similarly  $i_{\mathcal{G}'',0} = i_{\mathcal{H}_0,0} i_{\mathcal{G}''}$ .

**Theorem 8.6.** *The Redheffer coefficient-matrix symbol  $\left\{ \begin{bmatrix} s_0(n) & s_2(n) \\ s_1(n) & s(n) \end{bmatrix} \right\}_{n \in \mathbb{Z}}$  for a Lifting Problem can be recovered directly from the central extension  $\mathcal{U}_0$  (see (6.4) or (6.5) and (6.6)) according to the formula*

$$\begin{bmatrix} s_0(n) & s_2(n) \\ s_1(n) & s(n) \end{bmatrix} = \begin{bmatrix} i_{\mathcal{G}'',0}^* \\ i_{\tilde{\Delta},0}^* \mathcal{U}_0^* \end{bmatrix} \mathcal{U}_0^{*n} \begin{bmatrix} i_{\mathcal{G}',0} & i_{\tilde{\Delta}_*,0} \end{bmatrix}. \quad (8.40)$$

Moreover,

$$s_2(m) = 0 \text{ for } m \leq 0, \quad s_1(m) = 0 \text{ and } s(m) = 0 \text{ for } m < 0, \quad (8.41)$$

and also

$$s(0) = 0. \quad (8.42)$$

*Proof.* We first check that the formula (8.40) is correct for  $n < 0$ . By using the definitions (7.3) to unravel formula (7.6), we read off that, for  $n < 0$ ,

$$s_0(n) = i_{\mathcal{G}'',0}^* \mathcal{U}_0^{*n} i_{\mathcal{G}',0}, \quad s_1(n) = 0, \quad s_2(n) = 0, \quad s(n) = 0.$$

The first formula matches with the upper left corner of (8.40) for  $n < 0$ . As  $\mathcal{G}'' \subset \mathcal{K}''_-$  and  $\mathcal{G}' \subset \mathcal{K}'_+$  by assumption, the other three blocks match up for  $n < 0$  as a consequence of the identities (8.15), (8.16) and (8.17).

We next verify (8.40) for  $n \geq 0$ . From the Toeplitz structure (8.38) we see that the validity of (8.40) for  $n \geq 0$  is equivalent to showing that

$$\begin{bmatrix} s_0^+ & s_2^+ \\ s_1^+ & s^+ \end{bmatrix} : \begin{bmatrix} g' \\ i_{\tilde{\Delta}_*,0} \tilde{\delta}_* \end{bmatrix} \mapsto \left\{ \begin{bmatrix} i_{\mathcal{G}'',0}^* \\ i_{\tilde{\Delta},0}^* \mathcal{U}_0^{*n} \end{bmatrix} \mathcal{U}_0^* \begin{bmatrix} i_{\mathcal{G}',0} & i_{\tilde{\Delta}_*,0} \end{bmatrix} \begin{bmatrix} g' \\ i_{\tilde{\Delta}_*,0} \tilde{\delta}_* \end{bmatrix} \right\}_{n \in \mathbb{Z}_+}. \quad (8.43)$$

From the definition (7.3) we have

$$\begin{bmatrix} s_0^+ & s_2^+ \\ s_1^+ & s^+ \end{bmatrix} = \begin{bmatrix} \mathcal{I}_{\mathcal{G}''} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} S_0^+ & S_2^+ \\ S_1^+ & S^+ \end{bmatrix} \begin{bmatrix} i_{\mathcal{G}'} & 0 \\ 0 & I \end{bmatrix}.$$

Combining this with (8.43), we see that it suffices to show that

$$\begin{bmatrix} S_0^+ & S_2^+ i_{\tilde{\Delta}_*^{(0)}} \\ S_1^+ & S^+ i_{\tilde{\Delta}_*^{(0)}} \end{bmatrix} : \begin{bmatrix} h_0 \\ \tilde{\delta}_* \end{bmatrix} \mapsto \left\{ \begin{bmatrix} i_{\mathcal{H}_0,0}^* \\ i_{\tilde{\Delta},0}^* \mathcal{U}_0^* \end{bmatrix} \mathcal{U}_0^{*n} \begin{bmatrix} i_{\mathcal{G}',0} & i_{\tilde{\Delta}_*,0} \end{bmatrix} \begin{bmatrix} h_0 \\ \tilde{\delta}_* \end{bmatrix} \right\}_{n \in \mathbb{Z}_+}. \quad (8.44)$$

From the definition of  $\begin{bmatrix} S_0^+ & S_2^+ \\ S_1^+ & S^+ \end{bmatrix}$  as the forward-time augmented input-output map for the unitary colligation  $U_0$ , we know that

$$\begin{bmatrix} S_0^+ & S_2^+ i_{\tilde{\Delta}_*^{(0)}} \\ S_1^+ & S^+ i_{\tilde{\Delta}_*^{(0)}} \end{bmatrix} \begin{bmatrix} h_0 \\ \tilde{\delta}_* \end{bmatrix} = \left\{ \begin{bmatrix} h_0(n) \\ \tilde{\delta}(n) \end{bmatrix} \right\}_{n \in \mathbb{Z}_+}$$

means that

$$h_0(0) = h_0, \quad \begin{bmatrix} h_0(1) \\ \tilde{\delta}(0) \end{bmatrix} = U_0 \begin{bmatrix} h_0 \\ \tilde{\delta}_* \end{bmatrix}, \quad \begin{bmatrix} h_0(n+1) \\ \tilde{\delta}(n) \end{bmatrix} = U_0 \begin{bmatrix} h_0(n) \\ 0 \end{bmatrix} \text{ for } n > 0. \quad (8.45)$$

Given  $h_0 \in \mathcal{H}_0$  and  $\tilde{\delta}_* \in \tilde{\Delta}_*$ , let us define  $h_0(n) \in \mathcal{H}_0$  and  $\delta(n) \in \tilde{\Delta}$  by

$$\begin{bmatrix} h_0(n) \\ \delta(n) \end{bmatrix} = \begin{bmatrix} i_{\mathcal{H}_0,0}^* \\ i_{\tilde{\Delta},0}^* \end{bmatrix} \mathcal{U}_0^{*n} k \text{ where } k = i_{\mathcal{H}_0,0} h_0 + i_{\tilde{\Delta}_*,0} \tilde{\delta}_*. \quad (8.46)$$

Then (8.40) follows if we can show that  $\{h_0(n), \tilde{\delta}(n)\}_{n \in \mathbb{Z}_+}$  so defined satisfies (8.45).

The first equality in (8.45) is immediate from the fact that  $\text{im } i_{\mathcal{H}_0,0}$  is orthogonal to  $\text{im } i_{\tilde{\Delta}_*,0}$  in  $\mathcal{K}_0$ .

The second equality in (8.45) is an easy consequence of the general identity

$$U_0 = \begin{bmatrix} i_{\mathcal{H}_0,0}^* \\ i_{\tilde{\Delta},0}^* \end{bmatrix} \mathcal{U}_0^* \begin{bmatrix} i_{\mathcal{H}_0,0} & i_{\tilde{\Delta}_*,0} \end{bmatrix}, \quad (8.47)$$

connecting  $U_0$  and  $\mathcal{U}_0$ . This identity in turn is an easy consequence of the formula (8.3) for  $\mathcal{U}_0^*$  and is an analogue of the formula (6.3) connecting  $U_1$  and  $\mathcal{U}$  in a more general context.

The third equality in (8.45) can also be seen as a consequence of (8.47) as follows. For  $n > 0$  we compute

$$\begin{aligned} U_0 \begin{bmatrix} h_0(n) \\ 0 \end{bmatrix} &= U_0 \begin{bmatrix} i_{\mathcal{H}_0,0}^* \mathcal{U}_0^{*n} k \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} i_{\mathcal{H}_0,0}^* \\ i_{\tilde{\Delta},0}^* \end{bmatrix} \mathcal{U}_0^* i_{\mathcal{H}_0,0} i_{\tilde{\Delta}_*,0}^* \mathcal{U}_0^{*n} k \text{ (by (8.47))} \\ &= \begin{bmatrix} i_{\mathcal{H}_0,0}^* \\ i_{\tilde{\Delta},0}^* \end{bmatrix} \mathcal{U}_0^* P_{\mathcal{H}_0} \mathcal{U}_0^{*n} k \end{aligned} \quad (8.48)$$

where  $k = i_{\mathcal{H}_0,0} h_0 + i_{\tilde{\Delta}_*,0} \tilde{\delta}_*$  and where  $P_{\mathcal{H}_0}$  is the orthogonal projection of  $\mathcal{K}_0$  onto  $\text{im } i_{\mathcal{H}_0,0}$ . To verify the third equality in (8.45) it remains only to show that the projection  $P_{\mathcal{H}_0}$  is removable in the last expression in (8.48). To this end, use the orthogonal decomposition

$$\mathcal{K}_0 = \vec{i}_{\tilde{\Delta},0}(\ell_{\tilde{\Delta}}^2(\mathbb{Z}_-)) \oplus \text{im } i_{\mathcal{H}_0,0} \oplus \vec{i}_{\tilde{\Delta}_*,0}(\ell_{\tilde{\Delta}_*}^2(\mathbb{Z}_+)).$$

Note that  $\mathcal{U}_0^{*n} k \perp \vec{i}_{\tilde{\Delta}_*,0}(\ell_{\tilde{\Delta}_*}^2(\mathbb{Z}_+))$ , i.e.,

$$\mathcal{U}_0^{*n} k \in \vec{i}_{\tilde{\Delta},0}(\ell_{\tilde{\Delta}}^2(\mathbb{Z}_-)) \oplus \text{im } i_{\mathcal{H}_0,0} \quad (8.49)$$

for  $n > 0$ . Moreover it is easily checked

$$\mathcal{U}_0^* \vec{i}_{\tilde{\Delta},0}(\ell_{\tilde{\Delta}}^2(\mathbb{Z}_-)) \perp \text{im } i_{\tilde{\Delta},0} \oplus \text{im } i_{\mathcal{H}_0,0}. \quad (8.50)$$

From conditions (8.49) and (8.50) we see that indeed the projection  $P_{\mathcal{H}_0}$  is removable in (8.48) and the third equation in (8.45) follows as required.

Now that the validity of (8.40) is established, we see that  $s_2(0) = 0$  as a consequence of (8.15) for the case  $n = 0$ .

It remains to verify that  $s(0) = i_{\tilde{\Delta},0}^* \mathcal{U}_0 i_{\tilde{\Delta}_*,0} = 0$ , or equivalently,

$$\text{im } i_{\tilde{\Delta},0} \perp \mathcal{U}_0^* \text{im } i_{\tilde{\Delta}_*,0}.$$

This can be seen as a direct consequence of the definition of  $\mathcal{U}_0^*$  in (8.3). □



**Remark 8.7.** In the proof of Theorem 8.6 it is shown that, given that  $\mathcal{U}_0$  and  $U_0$  are related as in (8.47), then (8.46) implies (8.45). This observation can be seen as a special case of the following general result. *Given a unitary operator  $\mathcal{U}$  on  $\mathcal{K}$ , a unitary colligation  $U$  of the form*

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H} \\ \mathcal{E}_* \end{bmatrix}.$$

*such that*

$$U = \begin{bmatrix} i_{\mathcal{H}}^* \\ i_{\mathcal{E}_*}^* \end{bmatrix} \mathcal{U}^* \begin{bmatrix} i_{\mathcal{H}} & i_{\mathcal{E}} \end{bmatrix}. \quad (8.51)$$

*where*

$$i_{\mathcal{H}}: \mathcal{H} \rightarrow \mathcal{K}, \quad i_{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{K}, \quad i_{\mathcal{E}_*}: \mathcal{E}_* \rightarrow \mathcal{K}$$

*are isometric embedding operators with*

$$\text{im } i_{\mathcal{H}} \perp \text{im } i_{\mathcal{E}} \text{ so } \begin{bmatrix} i_{\mathcal{H}} & i_{\mathcal{E}} \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \rightarrow \mathcal{K} \text{ is isometric,}$$

$$\text{im } i_{\mathcal{H}} \perp \text{im } i_{\mathcal{E}_*} \text{ so } \begin{bmatrix} i_{\mathcal{H}} & i_{\mathcal{E}_*} \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E}_* \end{bmatrix} \rightarrow \mathcal{K} \text{ is isometric,}$$

*then, for any  $k \in \mathcal{K}$ , if  $(\vec{e}, \vec{h}, \vec{e}_*)$  of  $\ell_{\mathcal{E}}(\mathbb{Z}) \times \ell_{\mathcal{H}}(\mathbb{Z}) \times \ell_{\mathcal{E}_*}(\mathbb{Z})$  is given by*

$$\begin{aligned} e(n) &= i_{\mathcal{E}}^* \mathcal{U}^{*n} k, \\ h(n) &= i_{\mathcal{H}}^* \mathcal{U}^{*n} k, \\ e_*(n) &= i_{\mathcal{E}_*}^* \mathcal{U}^{*n+1} k, \end{aligned} \quad (8.52)$$

*then  $(\vec{e}, \vec{h}, \vec{e}_*)$  is a  $U$ -system trajectory, i.e., the system equations*

$$\begin{bmatrix} h(n+1) \\ e_*(n) \end{bmatrix} = U \begin{bmatrix} h(n) \\ e(n) \end{bmatrix} \quad (8.53)$$

*hold for all  $n \in \mathbb{Z}$ . Under these assumptions there is no a priori way to characterize which system trajectories  $(\vec{e}, \vec{h}, \vec{e}_*)$  arise from a  $k \in \mathcal{K}$  via formula (8.52). If we impose the additional structure:*

*$\text{im } i_{\mathcal{E}}$  and  $\text{im } i_{\mathcal{E}_*}$  are wandering subspaces for  $\mathcal{U}$ , so there exist uniquely determined isometric embedding operators*

$$\vec{i}_{\mathcal{E}}: \ell_{\mathcal{E}}^2(\mathbb{Z}) \rightarrow \mathcal{K}, \quad \vec{i}_{\mathcal{E}_*}: \ell_{\mathcal{E}_*}^2(\mathbb{Z}) \rightarrow \mathcal{K}$$

*which extend  $i_{\mathcal{E}}$  and  $i_{\mathcal{E}_*}$  in the sense that*

$$i_{\mathcal{E}} = \vec{i}_{\mathcal{E}} \circ i_{\mathcal{E}}^{(0)}, \quad i_{\mathcal{E}_*} = \vec{i}_{\mathcal{E}_*} \circ i_{\mathcal{E}_*}^{(-1)},$$

*and  $\mathcal{K}$  has the orthogonal decomposition*

$$\mathcal{K} = \text{im } i_{\mathcal{H}} \oplus \vec{i}_{\mathcal{E}_*}(\ell_{\mathcal{E}_*}^2(\mathbb{Z}_-)) \oplus \vec{i}_{\mathcal{E}}(\ell_{\mathcal{E}}^2(\mathbb{Z}_+)), \quad (8.54)$$

*then one can characterize the system trajectories of the form (8.52) as exactly those of finite-energy in the sense that*

$$\vec{e} \in \ell_{\mathcal{E}}^2(\mathbb{Z}) \text{ and } \vec{e}_* \in \ell_{\mathcal{E}_*}^2(\mathbb{Z}), \quad (8.55)$$

*or, equivalently, in the sense that*

$$\vec{e}|_{\mathbb{Z}_+} \in \ell_{\mathcal{E}}^2(\mathbb{Z}_+) \text{ and } \vec{e}_*|_{\mathbb{Z}_-} \in \ell_{\mathcal{E}_*}^2(\mathbb{Z}_-). \quad (8.56)$$

*With the additional wandering-subspace assumption and orthogonal-decomposition assumption (8.54) given above in place, then the map  $k \mapsto (e(n), h(n), e_*(n))$  defined*

by (8.52) gives a one-to-one correspondence between elements  $k$  of  $\mathcal{K}$  and finite-energy  $U$ -system trajectories  $(\vec{e}, \vec{h}, \vec{e}_*)$ . This last statement is essentially Lemma 2.3 in [10] and is the main ingredient in the coordinate-free approach in embedding a unitary colligation into a (discrete-time) Lax-Phillips scattering system. The reader can check that the situation in Theorem 8.6 meets all these assumptions (with  $\tilde{\Delta}$  in place of  $\mathcal{E}_*$  and  $\tilde{\Delta}_*$  in place of  $\mathcal{E}$ ); the computation in the proof of Theorem 8.6 exhibits the  $k = i_{\mathcal{H}_0} h_0 + i_{\tilde{\Delta}_*,0} \tilde{\delta}_*$  corresponding to the finite-energy system trajectory supported on  $\mathbb{Z}_+$  with initial condition  $h_0$  and impulse input supported at time  $n = 0$  equal to  $\tilde{\delta}_*$ . We invite the reader to consult [13] for an extension of these ideas to a several-variable context.

## 9. THE CHARACTERISTIC MEASURE OF THE UNIVERSAL SCATTERING SYSTEM AND ASSOCIATED HELLINGER-SPACE MODELS

In the sequel we assume that the subspaces  $\mathcal{G}' \subset \mathcal{K}'_+$  and  $\mathcal{G}'' \subset \mathcal{K}''_-$  are chosen to be

$$\mathcal{G}' = \mathcal{K}'_+, \quad \mathcal{G}'' = \mathcal{K}''_-.$$

Given the scattering system  $\mathfrak{S}_0$  (8.5) arising from the central extension  $\mathcal{U}_0$  associated with the data set

$$X, \quad (\mathcal{U}', \mathcal{K}'), \quad (\mathcal{U}'', \mathcal{K}''), \quad \mathcal{K}'_+ \subset \mathcal{K}', \quad \mathcal{K}''_- \subset \mathcal{K}'' \quad (9.1)$$

for a Lifting Problem, following the discussion in Section 3.1 (see (3.5)) with a minor adjustment, we define the *central characteristic measure*  $\hat{\Sigma}$  for the Lifting-Problem data set (9.1) by

$$\hat{\Sigma}_0(dt) = \begin{bmatrix} i_{\tilde{\Delta},0}^* \mathcal{U}_0^* \\ i_{\tilde{\Delta}_*,0}^* \\ i_{\mathcal{K}''_-,0}^* \\ i_{\mathcal{K}'_+,0}^* \end{bmatrix} E_{\mathcal{U}_0}(dt) \begin{bmatrix} \mathcal{U}_0 i_{\tilde{\Delta},0} & i_{\tilde{\Delta}_*,0} & i_{\mathcal{K}''_-,0} & i_{\mathcal{K}'_+,0} \end{bmatrix} \quad (9.2)$$

where  $E_{\mathcal{U}_0}(dt)$  is the spectral measure for the unitary operator  $\mathcal{U}_0$ . The next theorem lists some special properties of the central characteristic measure  $\hat{\Sigma}_0(dt)$ .

**Theorem 9.1.** *Suppose that  $\hat{\Sigma}_0(dt)$  is the central characteristic measure associated with a Lifting-Problem data set (9.1) as in (9.2). Then the following properties hold:*

- (1)  $\hat{\Sigma}_0$  is a positive operator measure of the form

$$\hat{\Sigma}_0 = \begin{bmatrix} m I_{\tilde{\Delta}} & \hat{s} & 0 & \hat{s}_1 \\ \hat{s}^* & m I_{\tilde{\Delta}_*} & \hat{s}_2^* & 0 \\ 0 & \hat{s}_2 & \sigma'' & \hat{s}_0 \\ \hat{s}_1^* & 0 & \hat{s}_0^* & \sigma' \end{bmatrix} \quad (9.3)$$

where  $m$  is Lebesgue measure on the unit circle  $\mathbb{T}$ .

- (2) The Redheffer coefficient-matrix symbol  $\left\{ \begin{bmatrix} s_0(m) & s_2(m) \\ s_1(m) & s(m) \end{bmatrix} \right\}_{m \in \mathbb{Z}}$  for the case  $\mathcal{G}' = \mathcal{K}'_+$ ,  $\mathcal{G}'' = \mathcal{K}''_-$  (see Theorem 7.1) is the moment sequence of the corresponding measures  $\hat{s}_0$ ,  $\hat{s}_2$ ,  $\hat{s}_1$  and  $\hat{s}$  appearing in  $\hat{\Sigma}_0$ :

$$\begin{bmatrix} s_0(m) & s_2(m) \\ s_1(m) & s(m) \end{bmatrix} = \begin{bmatrix} i_{\mathcal{K}''_-,0}^* \\ i_{\tilde{\Delta},0}^* \mathcal{U}_0^* \end{bmatrix} \mathcal{U}_0^{*m} \begin{bmatrix} i_{\tilde{\Delta}_*,0} & i_{\mathcal{K}'_+,0} \end{bmatrix} = \int_{\mathbb{T}} t^{-m} \begin{bmatrix} \hat{s}_0 & \hat{s}_2 \\ \hat{s}_1 & \hat{s} \end{bmatrix} (dt). \quad (9.4)$$

(3) The measures  $\widehat{s}$ ,  $\widehat{s}_2$ ,  $\widehat{s}_1$  are analytic operator-valued measures in the sense that

$$\int_{\mathbb{T}} t^{-m} \widehat{s}(dt) = 0 \text{ and } \int_{\mathbb{T}} t^{-m} \widehat{s}_1(dt) = 0 \text{ for } m < 0, \quad \int_{\mathbb{T}} t^{-m} \widehat{s}_2(dt) = 0 \text{ for } m \leq 0 \quad (9.5)$$

and moreover

$$\int_{\mathbb{T}} \widehat{s}(dt) = 0. \quad (9.6)$$

*Proof.* By the spectral theorem we see that the moment sequence of  $\widehat{\Sigma}_0$  is given by

$$\widehat{\Sigma}_{0,m} = \begin{bmatrix} i_{\widetilde{\Delta},0}^* \mathcal{U}_0^* \\ i_{\widetilde{\Delta}^*,0}^* \\ i_{\mathcal{K}'_-,0}^* \\ i_{\mathcal{K}'_+,0}^* \end{bmatrix} \mathcal{U}_0^{*m} \begin{bmatrix} \mathcal{U}_0 i_{\widetilde{\Delta},0} & i_{\widetilde{\Delta}^*,0} & i_{\mathcal{K}'_-,0} & i_{\mathcal{K}'_+,0}^* \end{bmatrix}.$$

In particular,

$$[\widehat{\Sigma}_{0,m}]_{1,1} = i_{\widetilde{\Delta},0}^* \mathcal{U}_0^{*m} i_{\widetilde{\Delta},0} = \delta_{m,0} I_{\widetilde{\Delta}}$$

(where  $\delta_{m,0}$  is the Kronecker delta symbol equal to 1 for  $m = 0$  and 0 for  $m \neq 0$ ) since  $\text{im } i_{\widetilde{\Delta},0}$  is wandering for  $\mathcal{U}_0$ . We conclude that the  $(1,1)$ -entry  $[\widehat{\Sigma}_0]_{1,1}$  of  $\widehat{\Sigma}_0$  is indeed  $mI_{\widetilde{\Delta}}$  where  $m$  is Lebesgue measure. Similarly, the  $(2,2)$ -entry  $[\widehat{\Sigma}_0]_{2,2}$  is equal to  $mI_{\widetilde{\Delta}^*}$ .

From (8.40) we read off that the Redheffer coefficient-matrix symbol (8.39) (for the case here where  $\mathcal{G}' = \mathcal{K}'_+$  and  $\mathcal{G}'' = \mathcal{K}''_-$ ) is given by

$$\begin{aligned} \begin{bmatrix} s_0(m) & s_2(m) \\ s_1(m) & s(m) \end{bmatrix} &= \begin{bmatrix} i_{\mathcal{K}''_-}^* \\ i_{\widetilde{\Delta},0}^* \mathcal{U}_0^* \end{bmatrix} \mathcal{U}_0^{*m} \begin{bmatrix} i_{\mathcal{K}'_+} & i_{\widetilde{\Delta}^*,0} \end{bmatrix} \\ &= \begin{bmatrix} [\widehat{\Sigma}_{0,m}]_{3,4} & [\widehat{\Sigma}_{0,m}]_{3,2} \\ [\widehat{\Sigma}_{0,m}]_{1,4} & [\widehat{\Sigma}_{0,m}]_{1,2} \end{bmatrix} \\ &= \begin{bmatrix} \widehat{s}_{0,m} & \widehat{s}_{2,m} \\ \widehat{s}_{1,m} & \widehat{s}_m \end{bmatrix} \\ &= \int_{\mathbb{T}} t^{-m} \begin{bmatrix} \widehat{s}_0 & \widehat{s}_2 \\ \widehat{s}_1 & \widehat{s} \end{bmatrix} (dt) \end{aligned}$$

from which (9.4) follows immediately. The analyticity properties (9.5) and (9.6) can now be seen as a consequence of the analyticity properties (8.41) and (8.42) of the Redheffer coefficient-matrix symbol, or, equivalently, as a consequence of the orthogonality relations (8.10) and the fact that  $\mathcal{U}_0^* \text{im } i_{\widetilde{\Delta}^*,0} \perp \text{im } i_{\widetilde{\Delta},0}$ .

It remains only to verify that  $[\widehat{\Sigma}_0]_{1,3} = 0$  and  $[\widehat{\Sigma}_0]_{2,4} = 0$ . In terms of moments, we must verify that

$$i_{\widetilde{\Delta},0}^* \mathcal{U}_0^{*m+1} i_{\mathcal{K}''_-,0} = 0 \text{ and } i_{\widetilde{\Delta}^*,0}^* \mathcal{U}_0^{*m} i_{\mathcal{K}'_+,0} = 0 \text{ for all } m \in \mathbb{Z}.$$

But these conditions are the strengthened versions (8.20) and (8.21) of the orthogonality relations (8.11) in Theorem 8.2. This concludes the proof of Theorem 9.1.  $\square$

We now use the central characteristic measure to provide a Hellinger space model for the central extension  $\mathcal{U}_0$  and the associated (slightly adjusted) scattering space  $\mathfrak{S}_0$  and four-fold AA-unitary coupling  $\mathfrak{S}_{0,AA}$  as follows. Consider the central scattering system with adjusted scale operator (still denoted  $\mathfrak{S}_0$ )

$$\left( \mathcal{U}_0, \begin{bmatrix} \mathcal{U}_0 i_{\tilde{\Delta},0} & i_{\tilde{\Delta}_*,0} & i_{\mathcal{K}''_-,0} & i_{\mathcal{K}'_+,0} \end{bmatrix}; \mathcal{K}_0, \tilde{\Delta} \oplus \tilde{\Delta}_* \oplus \mathcal{K}''_- \oplus \mathcal{K}'_+ \right).$$

As  $\text{im} \begin{bmatrix} i_{\mathcal{K}''_-,0} & i_{\mathcal{K}'_+,0} \end{bmatrix}$  is  $*$ -cyclic for  $\mathcal{U}_0$ , certainly  $\text{im} \begin{bmatrix} \mathcal{U}_0 i_{\tilde{\Delta},0} & i_{\tilde{\Delta}_*,0} & i_{\mathcal{K}''_-,0} & i_{\mathcal{K}'_+,0} \end{bmatrix}$  is  $*$ -cyclic for  $\mathcal{U}_0$  and the Fourier representation operator (see (3.17))

$$\mathcal{F}_0: k_0 \mapsto \begin{bmatrix} i_{\tilde{\Delta},0}^* \mathcal{U}_0^* \\ i_{\tilde{\Delta}_*,0}^* \\ i_{\mathcal{K}''_-,0}^* \\ i_{\mathcal{K}'_+,0}^* \end{bmatrix} E_{\mathcal{U}_0}(dt) k_0$$

maps  $\mathcal{K}_0$  unitarily onto the Hellinger space  $\mathcal{L}^{\widehat{\Sigma}_0}$  (see [18] for complete details). Inside  $\mathcal{L}^{\widehat{\Sigma}_0}$  we consider the following subspaces:

$$\begin{aligned} \mathcal{K}'' &:= \mathcal{F}_0(\text{im } i_{\mathcal{K}''_-,0}) = \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ 0 \\ \sigma''^{[-1]} \mathcal{L}^{\sigma''} \\ 0 \end{bmatrix}, \quad \mathcal{K}' &:= \mathcal{F}_0(\text{im } i_{\mathcal{K}'_+,0}) = \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sigma'^{[-1]} \mathcal{L}^{\sigma'} \end{bmatrix}, \\ \mathcal{K}''_- &:= \mathcal{F}_0(\text{im } i_{\mathcal{K}''_-,0}) = \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ 0 \\ \mathcal{K}''_- \\ 0 \end{bmatrix}, \quad \mathcal{K}'_+ &:= \mathcal{F}_0(\text{im } \mathcal{K}'_+) = \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathcal{K}'_+ \end{bmatrix}, \\ \mathcal{H}_0 &:= \mathcal{F}_0(\text{im } i_{\mathcal{H}_0,0}) = \text{clos } \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ 0 \\ \mathcal{K}''_- \\ \mathcal{K}'_+ \end{bmatrix}, \quad \mathcal{D} &:= \mathcal{F}_0(i_{\mathcal{H}_0,0}(\mathcal{D})) = \text{clos } \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ 0 \\ \mathcal{K}''_- \\ \mathcal{U}' \mathcal{K}'_+ \end{bmatrix}, \\ \mathcal{D}_* &:= \mathcal{F}_0(i_{\mathcal{H}_0,0}(\mathcal{D}_*)) = \text{clos } \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ 0 \\ \mathcal{U}''^* \mathcal{K}''_- \\ \mathcal{K}'_+ \end{bmatrix}, \\ \tilde{\Delta}_*^{(0)} &:= \mathcal{F}_0(\text{im } i_{\tilde{\Delta}_*,0}) = \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ \tilde{\Delta}_* \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{\Delta}^{(-1)} &:= \mathcal{F}_0(\text{im } i_{\tilde{\Delta},0}) = t^{-1} \widehat{\Sigma}_0 \begin{bmatrix} \tilde{\Delta} \\ 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned} \quad (9.7)$$

As a translation of the conditions (8.12) we see that then we also have

$$\begin{aligned} \widehat{\Sigma}_0 \begin{bmatrix} \tilde{\Delta} \\ 0 \\ 0 \\ 0 \end{bmatrix} &= \text{clos } \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ 0 \\ \mathcal{K}''_- \\ \mathcal{K}'_+ \end{bmatrix} \ominus \text{clos } \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ 0 \\ \mathcal{K}''_- \\ \mathcal{U}' \mathcal{K}'_+ \end{bmatrix}, \\ t^{-1} \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ \tilde{\Delta}_* \\ 0 \\ 0 \end{bmatrix} &= \text{clos } \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ 0 \\ \mathcal{K}''_- \\ \mathcal{K}'_+ \end{bmatrix} \ominus \text{clos } \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ 0 \\ \mathcal{U}''^* \mathcal{K}''_- \\ \mathcal{K}'_+ \end{bmatrix} \end{aligned} \quad (9.8)$$

and as a consequence of (8.13) we have

$$\mathcal{L}^{\widehat{\Sigma}_0} = \widehat{\Sigma}_0 \begin{bmatrix} H_{\widehat{\Delta}}^{2\perp} \\ 0 \\ 0 \\ 0 \end{bmatrix} \oplus \text{clos } \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ 0 \\ \mathcal{K}''_- \\ \mathcal{K}'_+ \end{bmatrix} \oplus \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ H_{\widehat{\Delta}_*}^2 \\ 0 \\ 0 \end{bmatrix}. \quad (9.9)$$

In addition we have the following result which generalizes a result of Adamjan-Arov-Kreĭn [2].

**Theorem 9.2.** *If  $\widehat{\Sigma}_0$  as in (9.2) is the central characteristic measure for a Lifting Problem, then*

$$\begin{bmatrix} mI_{\widehat{\Delta}} & \widehat{s} \\ \widehat{s}^* & mI_{\widehat{\Delta}_*} \end{bmatrix} = \begin{bmatrix} 0 & \widehat{s}_1 \\ \widehat{s}_2^* & 0 \end{bmatrix} \begin{bmatrix} \sigma'' & \widehat{s}_0 \\ \widehat{s}_0^* & \sigma' \end{bmatrix}^{[-1]} \begin{bmatrix} 0 & \widehat{s}_2 \\ \widehat{s}_1^* & 0 \end{bmatrix} \quad (9.10)$$

in the sense of measure Schur-complements as in [18].

*Proof.* We are given that  $\text{im } i_{\mathcal{K}''_-} + \text{im } i_{\mathcal{K}'_+}$  is  $*$ -cyclic for  $\mathcal{U}_0$  in  $\mathcal{K}_0$ . By transforming this condition to the space  $\mathcal{L}^{\widehat{\Sigma}_0}$  under the unitary transformation  $\mathcal{F}_0$ , we see that the space  $\mathcal{K}''_+ + \mathcal{K}''_-$  is weak- $*$  dense in  $\mathcal{L}^{\widehat{\Sigma}_0}$ . As a consequence of statement (3.18) in Theorem 3.7, we see that this is equivalent to the property that the Schur complement of the block  $\begin{bmatrix} \sigma'' & \widehat{s}_0 \\ \widehat{s}_0^* & \sigma' \end{bmatrix}$  in the matrix-measure  $\widehat{\Sigma}_0$  is zero, i.e., condition (9.10) holds.  $\square$

The next result says that, conversely, the properties (9.5), (9.6), (9.8) and (9.10) can be used to characterize central characteristic measures for a Lifting Problem.

**Theorem 9.3.** *Suppose that we are given unitary operators  $(\mathcal{U}'', \mathcal{K}'')$  and  $(\mathcal{U}', \mathcal{K}')$  together with  $*$ -cyclic subspaces  $\mathcal{K}''_- \subset \mathcal{K}''$  and  $\mathcal{K}'_+ \subset \mathcal{K}'$  invariant under  $\mathcal{U}''^*$  and  $\mathcal{U}'$  respectively. Suppose that  $\widetilde{\Delta}_*$  and  $\widetilde{\Delta}$  are two coefficient Hilbert spaces and that  $\widehat{\Sigma}_0$  is an  $\mathcal{L}(\widetilde{\Delta} \oplus \widetilde{\Delta}_* \oplus \mathcal{K}''_- \oplus \mathcal{K}'_+)$ -valued measure of the form (9.3) satisfying the following properties:*

(1) *The measures  $\sigma''$  and  $\sigma'$  are given by*

$$\sigma''(dt) = i_{\mathcal{K}''_- \rightarrow \mathcal{K}''}^* E_{\mathcal{U}''}(dt) i_{\mathcal{K}''_- \rightarrow \mathcal{K}''}, \quad \sigma'(dt) = i_{\mathcal{K}'_+ \rightarrow \mathcal{K}'}^* E_{\mathcal{U}'}(dt) i_{\mathcal{K}'_+ \rightarrow \mathcal{K}'} \quad (9.11)$$

*where  $i_{\mathcal{K}''_- \rightarrow \mathcal{K}''}$  and  $i_{\mathcal{K}'_+ \rightarrow \mathcal{K}'}$  are the inclusions of  $\mathcal{K}''_-$  into  $\mathcal{K}''$  and of  $\mathcal{K}'_+$  into  $\mathcal{K}'$  respectively, and where  $E_{\mathcal{U}''}$  and  $E_{\mathcal{U}'}$  are the spectral measures for  $\mathcal{U}''$  and  $\mathcal{U}'$  respectively.*

(2) *The measure Schur-complement condition (9.10) is satisfied.*

(3) *The measures  $\widehat{s}$ ,  $\widehat{s}_1$  and  $\widehat{s}_2$  satisfy the analyticity conditions (9.5).*

(4) *Conditions (9.8) hold.*

*Then there is a Lifting Problem so that  $(M_t, \mathcal{L}^{\widehat{\Sigma}_0})$  is the associated central lift, the collection*

$$\mathfrak{S}_0 = (M_t, \quad M_{\widehat{\Sigma}_0}; \quad \mathcal{L}^{\widehat{\Sigma}_0}, \quad \widetilde{\Delta} \oplus \widetilde{\Delta}_* \oplus \mathcal{K}''_- \oplus \mathcal{K}'_+) \quad (9.12)$$

*is the associated central scattering system, and*

$$\begin{bmatrix} s_0(m) & s_2(m) \\ s_1(m) & s(m) \end{bmatrix} := \int_{\mathbb{T}} t^{-m} \begin{bmatrix} \widehat{s}_0 & \widehat{s}_2 \\ \widehat{s}_1 & \widehat{s} \end{bmatrix} (dt)$$

is the associated Redheffer coefficient-matrix symbol. Here  $M_t$  is the operator of multiplication by the coordinate function  $t$  on  $\mathcal{L}^{\widehat{\Sigma}_0}$  and

$$M_{\widehat{\Sigma}_0} : \widetilde{\Delta} \oplus \widetilde{\Delta}_* \oplus \mathcal{K}''_- \oplus \mathcal{K}'_+ \rightarrow \mathcal{L}^{\widehat{\Sigma}_0}$$

is the model scale operator of multiplication on the left by the measure  $\widehat{\Sigma}_0$ .

**Remark 9.4.** Part of the content of Theorem 9.3 is that condition (9.6) is a consequence of the hypotheses given in the statement of the theorem, in particular, of the condition (9.8).

*Proof.* Let  $\widehat{\Sigma}_0$  be as in the statement of the theorem. If we ignore the middle expressions involving subspaces of  $\mathcal{K}_0$  and the Fourier representation operator  $\mathcal{F}_0$ , we may use formulas (9.7) to define subspaces  $\mathcal{K}''$ ,  $\mathcal{K}'$ ,  $\mathcal{K}''_-$ ,  $\mathcal{K}'_+$ ,  $\mathcal{H}_0$ ,  $\mathcal{D}$ ,  $\mathcal{D}_*$ ,  $\widetilde{\Delta}_*^{(0)}$ ,  $\widetilde{\Delta}_*^{(0)}$ ,  $\widetilde{\Delta}^{(-1)}$  of  $\mathcal{L}^{\widehat{\Sigma}_0}$ .

We wish to apply the first version of Theorem 8.4 to the case where  $\mathfrak{S}_0$  is equal to the Hellinger-model scattering system  $\mathfrak{S}_0$ . We first verify that the model scattering system  $\mathfrak{S}_0$  (9.12) extends to a model four-fold AA-unitary coupling

$$\mathfrak{S}_{AA,0} = (M_t, \quad \vec{\mathbf{i}}_{\widetilde{\Delta}}, \quad \vec{\mathbf{i}}_{\widetilde{\Delta}_*}, \quad \mathbf{i}_{\mathcal{K}''}, \quad \mathbf{i}_{\mathcal{K}'}; \quad \mathcal{L}^{\widehat{\Sigma}_0}) \quad (9.13)$$

of the unitary operators

$$(J, \ell_{\widetilde{\Delta}}^2(\mathbb{Z})), \quad (J, \ell_{\widetilde{\Delta}_*}^2(\mathbb{Z})), \quad (\mathcal{U}'', \mathcal{K}''), \quad (\mathcal{U}', \mathcal{K}') \quad (9.14)$$

as required in Theorem 8.4. For  $\vec{\delta} \in \ell_{\widetilde{\Delta}}^2(\mathbb{Z})$  of the form  $\vec{\delta} = J^n i_{\widetilde{\Delta}}^{(0)} \widetilde{\delta}$  for some  $n \in \mathbb{Z}$  and  $\widetilde{\delta} \in \widetilde{\Delta}$ , we define  $\vec{\mathbf{i}}_{\widetilde{\Delta}} \vec{\delta} = t^n \widehat{\Sigma}_0(dt) \begin{bmatrix} \widetilde{\delta} \\ 0 \\ 0 \end{bmatrix}$ . Since the  $(1,1)$ -entry of  $\widehat{\Sigma}_0$  is  $m \cdot I_{\widetilde{\Delta}}$ ,

it follows that  $\vec{\mathbf{i}}_{\widetilde{\Delta}}$  extends to a well-defined isometry mapping  $\ell_{\widetilde{\Delta}}^2(\mathbb{Z})$  into  $\mathcal{L}^{\widehat{\Sigma}_0}$  with the additional property that  $\mathbf{i}_{\widetilde{\Delta}} := t^{-1} \widehat{\Sigma}_0 \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} = \vec{\mathbf{i}}_{\widetilde{\Delta}} \circ i_{\widetilde{\Delta}}^{(-1)} : \widetilde{\Delta} \rightarrow \widetilde{\Delta}^{(-1)}$ . Similarly,

the formula  $\vec{\mathbf{i}}_{\widetilde{\Delta}_*} : J^n i_{\widetilde{\Delta}_*}^{(0)} \widetilde{\delta}_* \mapsto t^n \widehat{\Sigma}_0(dt) \begin{bmatrix} 0 \\ \widetilde{\delta}_* \\ 0 \end{bmatrix}$  extends to an isometric embedding of  $\ell_{\widetilde{\Delta}_*}^2(\mathbb{Z})$  into  $\mathcal{L}^{\widehat{\Sigma}_0}$  with the extension property  $\mathbf{i}_{\widetilde{\Delta}_*} := \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} = \vec{\mathbf{i}}_{\widetilde{\Delta}_*} \circ i_{\widetilde{\Delta}_*}^{(0)} : \widetilde{\Delta}_* \rightarrow \widetilde{\Delta}_*^{(0)}$ .

Moreover, the definition of  $\sigma'$  and  $\sigma''$  via (9.11) implies that  $M_t M_{\sigma'} = M_{\sigma'} \mathcal{U}'$  in  $\mathcal{L}(\mathcal{K}'_+, \mathcal{L}^{\sigma'})$  and  $M_{t^{-1}} M_{\sigma''} = M_{\sigma''} \mathcal{U}''^*$  in  $\mathcal{L}(\mathcal{K}''_-, \mathcal{L}^{\sigma''})$ . This implies that we can use the wave-operator construction to construct isometric embeddings

$$\mathbf{i}_{\mathcal{K}'} : \mathcal{K}' \rightarrow \mathcal{L}^{\widehat{\Sigma}_0}, \quad \mathbf{i}_{\mathcal{K}''} : \mathcal{K}'' \rightarrow \mathcal{L}^{\widehat{\Sigma}_0}$$

with the extension properties

$$\mathbf{i}_{\mathcal{K}'}|_{\mathcal{K}'_+} = \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ 0 \\ 0 \\ I_{\mathcal{K}'_+} \end{bmatrix} =: \mathbf{i}_{\mathcal{K}'_+} : \mathcal{K}'_+ \rightarrow \mathcal{K}'_+, \quad \mathbf{i}_{\mathcal{K}''}|_{\mathcal{K}''_-} = \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ 0 \\ I_{\mathcal{K}''_-} \\ 0 \end{bmatrix} =: \mathbf{i}_{\mathcal{K}''_-} : \mathcal{K}''_- \rightarrow \mathcal{K}''_-.$$

A consequence of the wave-operator construction is that

$$\text{im } \mathbf{i}_{\mathcal{K}''} = \mathcal{K}'', \quad \text{im } \mathbf{i}_{\mathcal{K}'} = \mathcal{K}'.$$

We also note the subspace identifications

$$\vec{\mathbf{i}}_{\widetilde{\Delta}}(\ell_{\widetilde{\Delta}}^2(\mathbb{Z}_-)) = \widehat{\Sigma}_0 \begin{bmatrix} H_{\widetilde{\Delta}}^{2+} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{\mathbf{i}}_{\widetilde{\Delta}_*}(\ell_{\widetilde{\Delta}_*}^2(\mathbb{Z}_+)) = \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ H_{\widetilde{\Delta}}^2 \\ 0 \\ 0 \end{bmatrix}.$$

It is now clear that (9.13) is a four-fold AA-unitary coupling of the four unitary operators (9.14) which extends the scattering system  $\mathfrak{S}_0$  as required in Theorem 8.4.

To apply the first version of Theorem 8.4, it remains to check the conditions (8.7), (8.10), (8.11) and (8.12). As noted in the proof of Theorem 9.2, the vanishing of the measure Schur-complement (9.10) is the functional-model version of the condition (8.7). The analyticity conditions (9.5) imply that

$$\mathcal{K}'_- \perp \widehat{\Sigma}_0 \begin{bmatrix} H_{\Delta}^{2\perp} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{K}'_+ \perp \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ H_{\Delta^*}^2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{K}''_- \perp \mathcal{K}'_+$$

where  $\mathcal{K}'_+$  and  $\mathcal{K}''_-$  are defined as in (9.7); one can check that these conditions are just the functional-model version of the orthogonality conditions (8.10). The presence of the 0 in the (1, 3) and (2, 4) locations of  $\widehat{\Sigma}_0$  is equivalent to the orthogonality conditions

$$\widehat{\Sigma}_0 \begin{bmatrix} L_{\Delta}^2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \perp \mathcal{K}'', \quad \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ L_{\Delta^*}^2 \\ 0 \\ 0 \end{bmatrix} \perp \mathcal{K}'$$

which is the functional-model equivalent of condition (8.11) in the stronger form (8.14). As we have already noted, the assumption (9.8) is the functional-model equivalent of conditions (8.12).

By Theorem 8.4 (first version) we conclude that  $(M_t, \mathcal{L}^{\widehat{\Sigma}_0})$  is the central extension,  $\mathfrak{S}_0$  is the central scattering system, and  $\mathfrak{S}_{AA,0}$  is the four-fold AA-unitary coupling associated with the Lifting Problem with data set

$$X = \mathbf{i}_{\mathcal{K}''}^* \mathbf{i}_{\mathcal{K}'} = \widehat{s}_0(\mathbb{T}), \quad (\mathcal{U}'', \mathcal{K}''), \quad (\mathcal{U}', \mathcal{K}'), \quad \mathcal{K}''_- \subset \mathcal{K}'', \quad \mathcal{K}'_+ \subset \mathcal{K}'. \quad (9.15)$$

Finally, application of the formula (8.40) to the present setting tells us that the associated Redheffer coefficient-matrix symbol is given by

$$\begin{aligned} \begin{bmatrix} s_0(m) & s_2(m) \\ s_1(m) & s(m) \end{bmatrix} &= \int_{\mathbb{T}} \begin{bmatrix} 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \end{bmatrix} t^{-m} \widehat{\Sigma}_0(dt) \begin{bmatrix} 0 & 0 \\ 0 & I \\ 0 & 0 \\ I & 0 \end{bmatrix} \\ &= \int_{\mathbb{T}} t^{-m} \begin{bmatrix} \widehat{s}_0 & \widehat{s}_2 \\ \widehat{s}_1 & \widehat{s} \end{bmatrix} (dt). \end{aligned}$$

This concludes the proof of Theorem 9.3.  $\square$

**Remark 9.5.** S. ter Horst [31] has obtained a solution of the inverse Relaxed Commutant Lifting (RCL) problem. Although, RCL is a generalization of the Commutant Lifting considered in the present paper, the setting of the inverse problem of [31] is quite different from our inverse problem even in the context of the classical Commutant Lifting. We discuss the difference in this remark.

If we use the formulation in Remark 7.3, we see that lifts  $Y$  have a representation of the form

$$Y = \begin{bmatrix} X \\ \Gamma_H D_X \end{bmatrix}$$

where  $\Gamma_H$  is the operator of multiplication by the function  $H(\zeta) := Y_{0+}(\zeta)$  as in formula (7.10). The set of such functions  $H(\zeta)$  is in turn parametrized by the Redheffer linear-fractional map  $H(\zeta) = \Phi_{11}(\zeta) + \Phi_{12}(\zeta)(I - V(\zeta)\Phi_{22}(\zeta))^{-1}\Phi_{21}(\zeta)$  where  $V$  is a free-parameter operator-valued Schur-class function of appropriate

size and  $\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}$  is the Redheffer coefficient matrix. The inverse problem as formulated in [31] is to characterize which Redheffer coefficient matrices arise in this way. The result is that *any* coefficient matrix  $\Phi: \mathbb{D} \rightarrow \mathcal{L}(\mathcal{U}_1 \oplus \mathcal{U}_2, \mathcal{Y}_1 \oplus \mathcal{Y}_2)$  such that the multiplication operator

$$\begin{bmatrix} M_{\Phi_{11}} & \Gamma_{\Phi_{12}} \\ M_{\Phi_{21}} & \Gamma_{\Phi_{22}} \end{bmatrix} : \begin{bmatrix} H_{\mathcal{U}_1}^2 \\ \mathcal{U}_2 \end{bmatrix} \rightarrow \begin{bmatrix} H_{\mathcal{Y}_1}^2 \\ H_{\mathcal{Y}_2}^2 \end{bmatrix}$$

is coisometric arises in this way. In this formulation of the inverse problem, there is flexibility in the construction of the underlying contraction  $\rho$ , i.e., a contraction  $\rho = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}$  so that the analogue of (7.15) is satisfied for all  $Y_{0+} = \mathcal{F}_\Psi[\omega]$ , as well as in the choice of a compatible RCL data set.

Our Theorems 8.4 and 9.3 can also be considered as solutions of inverse problems. In Theorem 9.3 we specify not only the Redheffer coefficient matrix but also the measures  $\sigma'$  and  $\sigma''$  giving Hellinger-space models for the unitary operators  $\mathcal{U}'$  and  $\mathcal{U}''$  as well as the subspaces  $\mathcal{K}'_+ \subset \mathcal{K}'$  and  $\mathcal{K}''_- \subset \mathcal{K}''$  and the intertwining operator  $X$ , i.e., the whole data set for the Lifting Problem. It is then automatic that the given Redheffer coefficient matrix parametrizes some subset of the set of all solutions of the associated Lifting Problem for this data set. The only remaining issue is to characterize the structure required which guarantees that the image of the Redheffer linear-fractional map gives rise to the set of *all* solutions of the given Lifting Problem. Theorem 8.4 is just a coordinate-free version of the same result: one specifies a four-fold AA-unitary coupling and seeks to characterize when it is the four-fold AA-unitary coupling coming from a Lifting Problem. Another version of the inverse theorem is Theorem 10.3 below where one is given just the Redheffer coefficient matrix along with the measures  $\sigma''$  and  $\sigma'$ .

We close this section with some observations concerning the absolute continuity of the measures  $\widehat{s}_1$ ,  $\widehat{s}_2$  and  $\widehat{s}$  inside the central characteristic measure coming from a Lifting Problem; these results generalize similar results for the case of the Nehari problem in [2].

**Theorem 9.6.** *Suppose that  $\widehat{\Sigma}_0$  is a positive strong operator-measure of the form (9.3). Then the measure  $\widehat{s}$  is uniformly absolutely continuous with respect to Lebesgue measure and the measures  $\widehat{s}_1$  and  $\widehat{s}_2$  are strongly absolutely continuous with respect to Lebesgue measure.*

*Proof.* The positivity of  $\widehat{\Sigma}_0$  (see (9.3)) implies that

$$\|\widehat{s}(B)\| \leq m(B)$$

for every Borel set  $B \subseteq \mathbb{T}$ ; this in turn means that  $\widehat{s}$  is uniformly absolutely continuous with respect to  $m$  (see Section 3 in [18] for more details about absolute continuity, in particular, definitions (3.1), (3.5), (3.23), (3.24) and formulas (3.6), (3.26) appearing there). The positivity of  $\widehat{\Sigma}_0$  also implies that

$$\|\widehat{s}_1(B)k'_+\| \leq \sqrt{m(B)} \sqrt{\langle \sigma'(B)k'_+, k'_+ \rangle}.$$

Then (see (3.1) of Section 3 in [18] for definition of the variation of a vector measure and also Theorem 3.4 *ibidem* for more details)

$$\text{var } \widehat{s}_1 k'_+(B) \leq \sqrt{m(B)} \sqrt{\langle \sigma'(B)k'_+, k'_+ \rangle}.$$



This means that the measure  $\widehat{s}_1 k'_+$  is absolutely continuous with respect to Lebesgue measure.  $\square$

**Remark 9.7.** Here we give a couple of alternate routes to results on the absolute continuity of  $\widehat{s}$ ,  $\widehat{s}_1$  and  $\widehat{s}_2$  which handle various special cases.

If we in addition impose the analyticity conditions (9.5) on the entries  $\widehat{s}_1$ ,  $\widehat{s}_2$  and  $\widehat{s}$ , then at least weak absolute continuity of  $\widehat{s}$ ,  $\widehat{s}_1$  and  $\widehat{s}_2$  with respect to Lebesgue measure is an immediate consequence of the F. and M. Riesz theorem (see e.g. [28, page 47]).

Alternatively, if we assume all the conditions in Theorem 9.3 so that  $\widehat{\Sigma}_0$  is the universal characteristic measure coming from a Lifting Problem, by using the connection (8.40) between  $\widehat{\Sigma}_0$  and the Redheffer coefficient-matrix symbol together with the explicit formula (see (7.3) and (7.6)) for the Redheffer coefficient matrix, we see that

$$\widehat{s}(dt) = s(t) \cdot m(dt), \quad \widehat{s}_1(t) = s_1(t) \cdot m(dt), \quad \widehat{s}_2(dt) = s_2(t) \cdot m(dt)$$

where  $s(t)$  is the boundary-value function for the Schur-class function

$$s(\zeta) = \zeta(i_{\mathcal{K}''_- \rightarrow \mathcal{H}_0})^* C_0 (I - \zeta A_0)^{-1} B_0 i_{\mathcal{K}'_+ \rightarrow \mathcal{H}_0}$$

and  $s_1(t)$  and  $s_2(t)$  are the boundary-value functions for the strong operator-valued  $H^2$ -functions

$$s_1(\zeta) = (i_{\mathcal{K}''_- \rightarrow \mathcal{H}_0})^* C_0 (I - \zeta A_0)^{-1} i_{\mathcal{K}'_+ \rightarrow \mathcal{H}_0}$$

$$s_2(\zeta) = (i_{\mathcal{K}''_- \rightarrow \mathcal{H}_0})^* (I - \zeta A_0)^{-1} B_0 i_{\mathcal{K}'_+ \rightarrow \mathcal{H}_0}$$

where  $U_0 = \begin{bmatrix} A_0 & B_0 \\ C_0^* & 0 \end{bmatrix}$  is the universal colligation constructed from the Lifting Problem data (see (6.4), (6.5), (6.6)).

## 10. A MORE COMPACT HELLINGER-SPACE MODEL AND MAXIMAL FACTORABLE MINORANTS

Suppose that  $\widehat{\Sigma}_0$  is the central characteristic measure (9.2) coming from a Lifting Problem. As we have seen, the subspace

$$\widehat{\Sigma}_0 \begin{bmatrix} 0 \\ 0 \\ \mathcal{K}''_-[t] \\ \mathcal{K}'_+[t^{-1}] \end{bmatrix} \quad (10.1)$$

(where  $\mathcal{K}''_-[t]$  is the space of analytic trigonometric polynomials with coefficients in  $\mathcal{K}''_-$  and similarly  $\mathcal{K}'_+[t^{-1}]$  is the space of conjugate-analytic trigonometric polynomials with coefficients from  $\mathcal{K}'_+$ ) is dense in  $\mathcal{L}^{\widehat{\Sigma}_0}$  as a consequence of the condition (8.7) (or, in measure-theoretic terms, of (9.10)). By applying Theorem 3.7 with  $\widehat{\sigma}_0 := \begin{bmatrix} \sigma'' & \widehat{s}_0 \\ \widehat{s}_0^* & \sigma' \end{bmatrix}$  in place of  $\sigma_{11}$ , we see that the closure of the space (10.1) can be identified with  $\mathcal{L}^{\widehat{\sigma}_0}$ . Moreover, as long as the condition (9.10) is in force, as a consequence of the observation (3.18) we see that the map

$$U_{\widehat{\sigma}_0, \widehat{\Sigma}_0} : \widehat{\sigma}_0 \begin{bmatrix} p'' \\ p' \end{bmatrix} \mapsto \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ 0 \\ p'' \\ p' \end{bmatrix} \quad (10.2)$$

(where  $\begin{bmatrix} p'' \\ p' \end{bmatrix} \in \begin{bmatrix} \mathcal{K}''_- \\ \mathcal{K}'_+ \end{bmatrix} [t, t^{-1}]$  is a trigonometric polynomial with coefficients in  $\begin{bmatrix} \mathcal{K}''_- \\ \mathcal{K}'_+ \end{bmatrix}$ ) extends to define a unitary transformation from  $\mathcal{L}^{\widehat{\sigma}_0}$  onto  $\mathcal{L}^{\widehat{\Sigma}_0}$ . Alternatively, we may construct directly a second model central scattering system with ambient space  $\mathcal{L}^{\widehat{\sigma}_0}$  as follows.

Define maps

$$\mathbf{i}_{\mathcal{K}'_+,0}: \mathcal{K}'_+ \rightarrow \mathcal{L}^{\widehat{\sigma}_0}, \quad \mathbf{i}_{\mathcal{K}''_-,0}: \mathcal{K}''_- \rightarrow \mathcal{L}^{\widehat{\sigma}_0}$$

by

$$\mathbf{i}_{\mathcal{K}'_+,0}: k'_+ \mapsto \widehat{\sigma}_0 \begin{bmatrix} 0 \\ k'_+ \end{bmatrix}, \quad \mathbf{i}_{\mathcal{K}''_-,0}: k''_- \mapsto \widehat{\sigma}_0 \begin{bmatrix} k''_- \\ 0 \end{bmatrix}$$

and extend them to

$$\mathbf{i}_{\mathcal{K}',0}: \mathcal{K}' \rightarrow \mathcal{L}^{\widehat{\sigma}_0}, \quad \mathbf{i}_{\mathcal{K}'',0}: \mathcal{K}'' \rightarrow \mathcal{L}^{\widehat{\sigma}_0}$$

by

$$\mathbf{i}_{\mathcal{K}',0}: \mathcal{U}'^{*n} k'_+ \mapsto \widehat{\sigma}_0 \begin{bmatrix} 0 \\ t^{-n} k'_+ \end{bmatrix}, \quad \mathbf{i}_{\mathcal{K}'',0}: \mathcal{U}''^{*n} k''_- \mapsto \widehat{\sigma}_0 \begin{bmatrix} t^n k''_- \\ 0 \end{bmatrix}$$

where  $k'_+ \in \mathcal{K}'_+$  and  $k''_- \in \mathcal{K}''_-$ . We also define subspaces

$$\widetilde{\Delta}_0^{(-1)} := \begin{bmatrix} 0 \\ t^{-1} \widehat{s}_1^* \widetilde{\Delta} \end{bmatrix}, \quad \widetilde{\Delta}_{*0}^{(0)} := \begin{bmatrix} \widehat{s}_2 \widetilde{\Delta}_* \\ 0 \end{bmatrix}$$

and define isometric embedding operators

$$\mathbf{i}_{\widetilde{\Delta},0}: \widetilde{\Delta} \rightarrow \widetilde{\Delta}_0^{(-1)}, \quad \mathbf{i}_{\widetilde{\Delta}_*,0}: \widetilde{\Delta}_* \rightarrow \widetilde{\Delta}_{*0}^{(0)}$$

by

$$\mathbf{i}_{\widetilde{\Delta},0}: \widetilde{\delta} \mapsto \begin{bmatrix} 0 \\ t^{-1} \widehat{s}_1^* \widetilde{\delta} \end{bmatrix}, \quad \mathbf{i}_{\widetilde{\Delta}_*,0}: \widetilde{\delta}_* \mapsto \begin{bmatrix} \widehat{s}_2 \widetilde{\delta}_* \\ 0 \end{bmatrix}$$

with extensions

$$\vec{\mathbf{i}}_{\widetilde{\Delta},0}: \ell_{\widetilde{\Delta}}^2(\mathbb{Z}) \rightarrow \mathcal{L}^{\widehat{\sigma}_0}, \quad \vec{\mathbf{i}}_{\widetilde{\Delta}_*,0}: \ell_{\widetilde{\Delta}_*}^2(\mathbb{Z}) \rightarrow \mathcal{L}^{\widehat{\sigma}_0}$$

given by

$$\begin{aligned} \vec{\mathbf{i}}_{\widetilde{\Delta},0}: \{\widetilde{\delta}(n)\}_{n \in \mathbb{Z}} &\mapsto \begin{bmatrix} 0 \\ \widehat{s}_1^* \left( \sum_{n \in \mathbb{Z}} \widetilde{\delta}(n) t^{n-1} \right) \end{bmatrix}, \\ \vec{\mathbf{i}}_{\widetilde{\Delta}_*,0}: \{\widetilde{\delta}_*(n)\}_{n \in \mathbb{Z}} &\mapsto \begin{bmatrix} \widehat{s}_2 \left( \sum_{n \in \mathbb{Z}} \widetilde{\delta}_*(n) t^n \right) \\ 0 \end{bmatrix}. \end{aligned}$$

The fact that all these maps are isometries is a consequence of the identity (9.10). One can check (with occasional use of various tools for manipulation of Hellinger spaces from [18] to complete the details) that

$$\mathfrak{S}_{00} := (M_t, \quad [\mathbf{i}_{\widetilde{\Delta},0} \quad \mathbf{i}_{\widetilde{\Delta}_*,0} \quad \mathbf{i}_{\mathcal{K}''_-,0} \quad \mathbf{i}_{\mathcal{K}'_+,0}]; \quad \mathcal{L}^{\widehat{\sigma}_0}, \quad \widetilde{\Delta} \oplus \widetilde{\Delta}_* \oplus \mathcal{K}''_- \oplus \mathcal{K}'_+)$$

is also a central scattering system associated with the same Lifting-Problem data set (9.1) which extends (in the sense of Theorem 8.4) to the four-fold AA-unitary coupling

$$\mathfrak{S}_{AA,00} := (M_t, \quad \vec{\mathbf{i}}_{\widetilde{\Delta},0}, \quad \vec{\mathbf{i}}_{\widetilde{\Delta}_*,0}, \quad \mathbf{i}_{\mathcal{K}''_-,0}, \quad \mathbf{i}_{\mathcal{K}'_+,0}; \quad \mathcal{L}^{\widehat{\sigma}_0})$$

of the unitary operators (9.14). We mention that the property (8.8) assumes the following form for the  $\mathcal{L}^{\widehat{\sigma}_0}$  functional model:

$$\begin{aligned} \text{clos } \widehat{\sigma}_0 \begin{bmatrix} \mathcal{K}''_-[t] \\ \mathcal{K}'_+ \end{bmatrix} &= \text{im } \mathbf{i}_{\mathcal{H}_{0,0}} \oplus \begin{bmatrix} \widehat{s}_2 H_{\Delta_*}^2 \\ 0 \end{bmatrix} = \mathcal{L}^{\widehat{\sigma}_0} \ominus \begin{bmatrix} 0 \\ \widehat{s}_1^* H_{\Delta}^{2\perp} \end{bmatrix}, \\ \text{clos } \widehat{\sigma}_0 \begin{bmatrix} \mathcal{K}''_- \\ \mathcal{K}'_+[t^{-1}] \end{bmatrix} &= \text{im } \mathbf{i}_{\mathcal{H}_{0,0}} \oplus \begin{bmatrix} 0 \\ \widehat{s}_1^* H_{\Delta}^{2\perp} \end{bmatrix} = \mathcal{L}^{\widehat{\sigma}_0} \ominus \begin{bmatrix} \widehat{s}_2 H_{\Delta_*}^2 \\ 0 \end{bmatrix}. \end{aligned} \quad (10.3)$$

We note that construction of the space  $\mathcal{L}^{\widehat{\sigma}_0}$  does not explicitly make use of  $\widehat{s}_1$ ,  $\widehat{s}_2$ ,  $\widehat{s}$  which appear in  $\widehat{\Sigma}_0$ ; hence it must be the case that  $\widehat{s}_1$ ,  $\widehat{s}_2$  and  $\widehat{s}$  are already somehow encoded in the ingredients  $\sigma'$ ,  $\sigma''$  and  $\widehat{s}_0$  of  $\widehat{\Sigma}_0$ . The goal of this final section is to make this idea precise. The first step is the following result concerning maximal factorable minorants.

**Theorem 10.1.** *Suppose that  $\widehat{\Sigma}_0$  of the form (9.3) is the central characteristic measure (9.2) for some Lifting Problem. Then the following hold:*

(1) *The domination property*

$$\sigma' - \widehat{s}_0^* \sigma''^{[-1]} \widehat{s}_0 \geq \widehat{s}_1^* \frac{1}{m} \widehat{s}_1 \quad (10.4)$$

*holds between  $\widehat{s}_0$  and  $\widehat{s}_1$ . Moreover,  $\widehat{s}_1^* \frac{1}{m} \widehat{s}_1$  is a right maximal factorable minorant of  $\sigma' - \widehat{s}_0^* \sigma''^{[-1]} \widehat{s}_0$  in the following sense: if  $r_1^*$  is a strong  $\mathcal{L}(\mathcal{K}'_+, \mathcal{N})$ -valued conjugate-analytic measure such that*

$$\sigma' - \widehat{s}_0^* \sigma''^{[-1]} \widehat{s}_0 \geq r_1^* \frac{1}{m} r_1, \quad (10.5)$$

*then there is a contractive strongly conjugate-analytic  $\mathcal{L}(\mathcal{N}, \widetilde{\Delta})$ -valued function  $\theta_1^*$  such that*

$$r_1^* = \widehat{s}_1^* \theta_1^* \quad (10.6)$$

*(where the equality between measures holds in the strong sense).*

(2) *The domination property*

$$\sigma'' - \widehat{s}_0 \sigma'^{[-1]} \widehat{s}_0^* \geq (t^{-1} \widehat{s}_2) \frac{1}{m} (t^{-1} \widehat{s}_2)^* \quad (10.7)$$

*holds between  $\widehat{s}_0$  and  $\widehat{s}_2$ . Moreover,  $(t^{-1} \widehat{s}_2) \frac{1}{m} (t^{-1} \widehat{s}_2)^*$  is a left maximal factorable minorant of  $\sigma'' - \widehat{s}_0 \sigma'^{[-1]} \widehat{s}_0^*$  in the following sense: if  $r_2$  is a strong  $\mathcal{L}(\mathcal{N}_*, \mathcal{K}''_-)$ -valued analytic measure such that*

$$\sigma'' - \widehat{s}_0 \sigma'^{[-1]} \widehat{s}_0^* \geq r_2 \frac{1}{m} r_2^*, \quad (10.8)$$

*then there is a contractive strongly analytic  $\mathcal{L}(\mathcal{N}_*, \widetilde{\Delta}_*)$ -valued function  $\theta_2$  such that*

$$r_2 = (t^{-1} \widehat{s}_2) \theta_2. \quad (10.9)$$

*Thus, given a measure  $\widehat{s}_0$  so that  $\begin{bmatrix} \sigma'' & \widehat{s}_0 \\ \widehat{s}_0^* & \sigma' \end{bmatrix} \geq 0$ , the remaining nonzero entries  $\widehat{s}_1$  and  $\widehat{s}_2$  of a central measure  $\widehat{\Sigma}_0$  for a lifting problem are uniquely determined up to unitary-constant left/right-factor normalizations of  $\widehat{s}_1$  and  $\widehat{s}_2$ .*

*Proof.* We prove only the first statement as the second is completely analogous. From the positivity of  $\widehat{\Sigma}_0$  we deduce the positivity of any of the  $3 \times 3$  principal submatrices, in particular:

$$\begin{bmatrix} mI_{\widetilde{\Delta}} & 0 & \widehat{s}_1 \\ 0 & \sigma'' & \widehat{s}_0 \\ \widehat{s}_1^* & \widehat{s}_0^* & \sigma' \end{bmatrix} \geq 0. \quad (10.10)$$

By a standard Schur-complement argument, this is equivalent to

$$\sigma' - \begin{bmatrix} \widehat{s}_1^* & \widehat{s}_0^* \end{bmatrix} \begin{bmatrix} mI_{\widetilde{\Delta}} & 0 \\ 0 & \sigma'' \end{bmatrix}^{[-1]} \begin{bmatrix} \widehat{s}_1 \\ \widehat{s}_0 \end{bmatrix} \geq 0$$

from which we get (10.4).

Next, suppose that  $r_1$  is as in the hypotheses of the theorem. By the same Schur-complement argument one can see that (10.5) is equivalent to

$$\widehat{\sigma}_0 := \begin{bmatrix} \sigma'' & \widehat{s}_0 \\ \widehat{s}_0^* & \sigma' \end{bmatrix} \geq \begin{bmatrix} 0 \\ r_1^* \end{bmatrix} \frac{1}{m} \begin{bmatrix} 0 & r \end{bmatrix}.$$

Another Schur-complement argument converts this to

$$\begin{bmatrix} mI_{\mathcal{N}} & 0 & r_1 \\ 0 & \sigma'' & \widehat{s}_0 \\ r_1^* & \widehat{s}_0^* & \sigma' \end{bmatrix} \geq 0.$$

By the definition of Hellinger spaces (see Section 3.2 and especially (3.10)), it now follows that  $\begin{bmatrix} 0 \\ r_1^* \end{bmatrix} n \in \mathcal{L}^{\widehat{\sigma}_0}$  for each  $n \in \mathcal{N}$ . Due to the conjugate-analyticity of  $r_1^* n$  we then have that

$$t^{-1} \begin{bmatrix} 0 \\ r_1^* n \end{bmatrix} \perp \widehat{\sigma}_0 \begin{bmatrix} \mathcal{K}''_{-}[t] \\ \mathcal{K}'_{+} \end{bmatrix}. \quad (10.11)$$

As a consequences of (10.3), the closure of the space  $\widehat{\sigma}_0 \begin{bmatrix} \mathcal{K}''_{-}[t] \\ \mathcal{K}'_{+} \end{bmatrix}$  can be identified exactly as the orthogonal complement of the space  $\begin{bmatrix} 0 \\ \widehat{s}_1^* H_{\widetilde{\Delta}}^{2\perp} \end{bmatrix}$  in  $\mathcal{L}^{\widehat{\sigma}_0}$ . We conclude that there is an element  $g_{-,n}$  of  $H_{\widetilde{\Delta}}^{2\perp}$  so that

$$t^{-1} \begin{bmatrix} 0 \\ r_1^* n \end{bmatrix} = \begin{bmatrix} 0 \\ \widehat{s}_1^* \end{bmatrix} g_{-,n}.$$

As the correspondence  $n \mapsto g_{-,n}$  is linear, it follows that there is a strongly conjugate-analytic function  $\theta_1^*$  so that  $g_{-,n} = t^{-1} \theta_1^* n$ . From (10.11) we now arrive at (10.6). Taking the Schur complement of  $\begin{bmatrix} \sigma'' & \widehat{s}_0 \\ \widehat{s}_0^* & \sigma' \end{bmatrix}$  in (10.10) then gives

$$mI_{\mathcal{N}} - \begin{bmatrix} 0 & r_1 \end{bmatrix} \begin{bmatrix} \sigma'' & \widehat{s}_0 \\ \widehat{s}_0^* & \sigma' \end{bmatrix}^{[-1]} \begin{bmatrix} 0 \\ r_1^* \end{bmatrix} = mI_{\mathcal{N}} - \theta_1 \begin{bmatrix} 0 & \widehat{s}_1 \end{bmatrix} \begin{bmatrix} \sigma'' & \widehat{s}_0 \\ \widehat{s}_1^* & \sigma' \end{bmatrix}^{[-1]} \begin{bmatrix} 0 \\ \widehat{s}_1^* \end{bmatrix} \theta_1^* \geq 0.$$

But a consequence of (9.10) is that

$$\begin{bmatrix} 0 & \widehat{s}_1 \end{bmatrix} \begin{bmatrix} \sigma'' & \widehat{s}_0 \\ \widehat{s}_0^* & \sigma' \end{bmatrix}^{[-1]} \begin{bmatrix} 0 \\ \widehat{s}_1^* \end{bmatrix} = mI_{\widetilde{\Delta}}$$

and it follows that  $I_{\mathcal{N}} - \theta_1 \theta_1^* \geq 0$  and  $\theta$  is contractive as required. This completes the proof of Theorem 10.1.  $\square$

In a similar vein one can obtain another positivity property for which the maximal factorable minorant is zero.

**Theorem 10.2.** Suppose that  $\widehat{\Sigma}_0$  of the form (9.3) is the central characteristic measure (9.2) for some Lifting Problem. Let

$$\widehat{S} = \begin{bmatrix} \widehat{s}_0 & \widehat{s}_2 \\ \widehat{s}_1 & \widehat{s} \end{bmatrix}$$

be the Fourier transform of the associated Redheffer coefficient-matrix symbol. Then the following hold true:

(1)

$$\begin{bmatrix} \sigma'' & 0 \\ 0 & mI_{\widetilde{\Delta}^*} \end{bmatrix} - \widehat{S}^* \begin{bmatrix} \sigma' & 0 \\ 0 & mI_{\widetilde{\Delta}^*} \end{bmatrix}^{[-1]} \widehat{S} \geq 0. \quad (10.12)$$

Moreover, 0 is the right maximal factorable minorant for (10.12) in the following sense: if  $\Phi^*$  is a strongly conjugate-analytic  $\mathcal{L}\left(\mathcal{N}, \begin{bmatrix} \kappa'_+ \\ \widetilde{\Delta}^* \end{bmatrix}\right)$ -valued measure so that

$$\begin{bmatrix} \sigma'' & 0 \\ 0 & mI_{\widetilde{\Delta}^*} \end{bmatrix} - \widehat{S}^* \begin{bmatrix} \sigma' & 0 \\ 0 & mI_{\widetilde{\Delta}^*} \end{bmatrix}^{[-1]} \widehat{S} \geq \Phi^* \frac{1}{m} \Phi, \quad (10.13)$$

then  $\Phi = 0$ .

(2)

$$\begin{bmatrix} \sigma'' & 0 \\ 0 & mI_{\widetilde{\Delta}^*} \end{bmatrix} - \widehat{S} \begin{bmatrix} \sigma'' & 0 \\ 0 & mI_{\widetilde{\Delta}} \end{bmatrix}^{[-1]} \widehat{S}^* \geq 0. \quad (10.14)$$

Moreover, 0 is the left maximal factorable minorant for (10.14) in the following sense: if  $\Phi_*$  is a strongly analytic  $\mathcal{L}\left(\mathcal{N}_*, \begin{bmatrix} \kappa''_- \\ \widetilde{\Delta} \end{bmatrix}\right)$ -valued measure such that

$$\begin{bmatrix} \sigma'' & 0 \\ 0 & mI_{\widetilde{\Delta}^*} \end{bmatrix} - \widehat{S} \begin{bmatrix} \sigma'' & 0 \\ 0 & mI_{\widetilde{\Delta}} \end{bmatrix}^{[-1]} \widehat{S}^* \geq \Phi_* \frac{1}{m} \Phi_*,$$

then  $\Phi_* = 0$ .

*Proof.* We prove only part (1) as part (2) is similar. After interchanging the second and third rows and then the second and third columns in (9.3), we get

$$\widetilde{\Sigma}'_0 := \begin{bmatrix} \begin{bmatrix} mI_{\widetilde{\Delta}} & 0 \\ 0 & \sigma'' \end{bmatrix} & \widehat{S}' \\ \widehat{S}'^* & \begin{bmatrix} mI_{\widetilde{\Delta}^*} & 0 \\ 0 & \sigma' \end{bmatrix} \end{bmatrix} := \begin{bmatrix} mI_{\widetilde{\Delta}} & 0 & \widehat{s} & \widehat{s}_1 \\ 0 & \sigma'' & \widehat{s}_2 & \widehat{s}_0 \\ \widehat{s}^* & \widehat{s}_2^* & mI_{\widetilde{\Delta}^*} & 0 \\ \widehat{s}_1^* & \widehat{s}_0^* & 0 & \sigma' \end{bmatrix} \geq 0,$$

If we then interchange the first two rows and then the first two columns and then the last two rows followed by the last two columns, we arrive at

$$\widetilde{\Sigma}_0 = \begin{bmatrix} \begin{bmatrix} \sigma'' & 0 \\ 0 & mI_{\widetilde{\Delta}} \end{bmatrix} & \widehat{S} \\ \widehat{S}^* & \begin{bmatrix} \sigma' & 0 \\ 0 & mI_{\widetilde{\Delta}^*} \end{bmatrix} \end{bmatrix} := \begin{bmatrix} \sigma'' & 0 & \widehat{s}_0 & \widehat{s}_2 \\ 0 & mI_{\widetilde{\Delta}} & \widehat{s}_1 & \widehat{s} \\ \widehat{s}_0^* & \widehat{s}_1^* & \sigma' & 0 \\ \widehat{s}_2^* & \widehat{s}^* & 0 & mI_{\widetilde{\Delta}^*} \end{bmatrix}.$$

Clearly, the positivity of  $\widetilde{\Sigma}_0$  is equivalent to the positivity of  $\widehat{\Sigma}_0$ . By a standard Schur-complement argument, the positivity of  $\widetilde{\Sigma}_0$  in turn is equivalent to (10.12).

Clearly, 0 is a left maximal factorable minorant for (10.12) if and only if 0 is a maximal factorable minorant for the Schur complement in  $\tilde{\Sigma}'_0$ :

$$\begin{bmatrix} mI_{\tilde{\Delta}^*} & 0 \\ 0 & \sigma' \end{bmatrix} - \hat{S}'^* \begin{bmatrix} mI_{\tilde{\Delta}} & 0 \\ 0 & \sigma'' \end{bmatrix}^{[-1]} \hat{S}' \geq 0. \quad (10.15)$$

By a Schur-complement argument, if in fact (10.13) holds for a strongly conjugate-analytic  $\Phi^*$ , then

$$\begin{bmatrix} mI_{\mathcal{N}} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \Phi \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} mI_{\tilde{\Delta}} & 0 \\ 0 & \sigma'' \end{bmatrix} & \hat{S}' \\ \Phi^* & \hat{S}'^* & \begin{bmatrix} mI_{\tilde{\Delta}^*} & 0 \\ 0 & \sigma' \end{bmatrix} \end{bmatrix} \geq 0.$$

By the definition of the Hellinger space, this in turn implies that  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\Phi^*} n \in \mathcal{L}^{\tilde{\Sigma}'_0}$  for every  $n \in \mathcal{N}$ . Since by assumption  $\Phi^*$  is strongly conjugate-analytic, then, as in the proof of Theorem 10.1, we have

$$t^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\Phi^* n} \perp \begin{bmatrix} \begin{bmatrix} mI_{\tilde{\Delta}} & 0 \\ 0 & \sigma'' \end{bmatrix} & \hat{S}' \\ \hat{S}'^* & \begin{bmatrix} mI_{\tilde{\Delta}^*} & 0 \\ 0 & \sigma' \end{bmatrix} \end{bmatrix} \begin{bmatrix} L_{\tilde{\Delta}}^2 \\ \mathcal{K}''_{-}[t] \\ H_{\tilde{\Delta}^*}^2 \\ \mathcal{K}'_{+} \end{bmatrix}.$$

But by (9.9), we know that the linear manifold

$$\begin{bmatrix} \begin{bmatrix} mI_{\tilde{\Delta}} & 0 \\ 0 & \sigma'' \end{bmatrix} & \hat{S}' \\ \hat{S}'^* & \begin{bmatrix} mI_{\tilde{\Delta}^*} & 0 \\ 0 & \sigma' \end{bmatrix} \end{bmatrix} \begin{bmatrix} H_{\tilde{\Delta}}^{2\perp} \\ \mathcal{K}''_{-} \\ H_{\tilde{\Delta}^*}^2 \\ \mathcal{K}'_{+} \end{bmatrix}$$

is already dense in  $\mathcal{L}^{\tilde{\Sigma}'_0}$ . Therefore necessarily  $\Phi = 0$ , completing the promised proof of part (1) of the Theorem.  $\square$

We next show that Theorem 10.1 leads to the following characterization of central characteristic measures; this version is more intrinsically function-theoretic than the characterization given by Theorem 9.3.

**Theorem 10.3.** *Suppose that  $\hat{\Sigma}_0$  is a strong  $\mathcal{L}(\tilde{\Delta} \oplus \tilde{\Delta}^* \oplus \mathcal{K}''_{-} \oplus \mathcal{K}'_{+})$ -valued measure of the form (9.3). Suppose that  $\hat{\sigma}_0 := \begin{bmatrix} \sigma'' & \hat{s}_0 \\ \hat{s}_0^* & \sigma' \end{bmatrix}$  is a positive  $\mathcal{L}(\mathcal{K}''_{-} \oplus \mathcal{K}'_{+})$ -valued measure which determines the remaining entries  $\hat{s}_1, \hat{s}_2, \hat{s}$  in  $\hat{\Sigma}_0$  according to the following procedure:*

- (1)  $\hat{s}_1$  is an analytic measure such that  $\hat{s}_1^* \frac{1}{m} \hat{s}_1$  is a maximal right factorable minorant for  $\sigma' - \hat{s}_0^* \sigma''^{[-1]} \hat{s}_0$ .
- (2)  $\hat{s}_2$  is an analytic measure with  $\hat{s}_2(\mathbb{T}) = 0$  such that  $(t^{-1} \hat{s}_2) \frac{1}{m} (t^{-1} \hat{s}_2)^*$  is a maximal left factorable minorant for  $\sigma'' - \hat{s}_0 \sigma'^{[-1]} \hat{s}_0^*$ .
- (3) The formula

$$\begin{bmatrix} mI_{\tilde{\Delta}} & \hat{s} \\ \hat{s}^* & mI_{\tilde{\Delta}^*} \end{bmatrix} = \begin{bmatrix} 0 & \hat{s}_1 \\ \hat{s}_2^* & 0 \end{bmatrix} \begin{bmatrix} \sigma'' & \hat{s}_0 \\ \hat{s}_0^* & \sigma' \end{bmatrix}^{[-1]} \begin{bmatrix} 0 & \hat{s}_2 \\ \hat{s}_1^* & 0 \end{bmatrix} \quad (10.16)$$

holds true for the analytic measure  $\widehat{s}$  which in addition satisfies

$$\widehat{s}(\mathbb{T}) = \int_{\mathbb{T}} \widehat{s}(dt) = 0. \quad (10.17)$$

Assume in addition that

$$\left( \mathcal{L}^{\widehat{\sigma}_0} - \text{clos } \widehat{\sigma}_0 \begin{bmatrix} \mathcal{K}''_-[t] \\ \mathcal{K}'_+ \end{bmatrix} \right) \cap \left( \mathcal{L}^{\widehat{\sigma}_0} - \text{clos } \widehat{\sigma}_0 \begin{bmatrix} \mathcal{K}''_- \\ \mathcal{K}'_+[t^{-1}] \end{bmatrix} \right) = \mathcal{L}^{\widehat{\sigma}_0} - \text{clos } \widehat{\sigma}_0 \begin{bmatrix} \mathcal{K}''_- \\ \mathcal{K}'_+ \end{bmatrix}. \quad (10.18)$$

Then  $\widehat{\Sigma}_0$  is the central characteristic measure arising from some Lifting Problem.

*Proof.* Assume that  $\widehat{\Sigma}_0$  has the form (9.3) with  $\widehat{s}_1$ ,  $\widehat{s}_2$  and  $\widehat{s}_0$  determined from  $\widehat{\sigma}_0 = \begin{bmatrix} \sigma'' & \widehat{s}_0 \\ \widehat{s}_0^* & \sigma' \end{bmatrix}$  as in the statement of the theorem. We wish to apply the second version of Theorem 8.4 to conclude that  $\widehat{\Sigma}^0$  is the central characteristic measure for a Lifting Problem.

From the hypothesis (10.16) combined with the observation (3.18), we see that the first hypothesis (8.7) required for application of the second version of Theorem 8.4 holds.

The maximal-factorable-minorant properties of  $\widehat{s}_1$  and  $\widehat{s}_2$  imply that

$$\begin{aligned} \text{clos } \widehat{\sigma}_0 \begin{bmatrix} \mathcal{K}''_-[t] \\ \mathcal{K}'_+ \end{bmatrix} &= \mathcal{L}^{\widehat{\sigma}_0} \ominus \begin{bmatrix} 0 \\ \widehat{s}_1 H_{\Delta^*}^{2\perp} \end{bmatrix}, \\ \text{clos } \widehat{\sigma}_0 \begin{bmatrix} \mathcal{K}''_- \\ \mathcal{K}'_+[t^{-1}] \end{bmatrix} &= \mathcal{L}^{\widehat{\sigma}_0} \ominus \begin{bmatrix} \widehat{s}_2 H_{\Delta}^2 \\ 0 \end{bmatrix}. \end{aligned} \quad (10.19)$$

Property (10.16) implies that the map  $U_{\widehat{\sigma}_0, \widehat{\Sigma}_0}$  given by (10.2) is unitary from  $\mathcal{L}^{\widehat{\sigma}_0}$  onto  $\mathcal{L}^{\widehat{\Sigma}_0}$ . One can check that

$$U_{\widehat{\sigma}_0, \widehat{\Sigma}_0} : \begin{bmatrix} 0 \\ \widehat{s}_1 H_{\Delta^*}^{2\perp} \end{bmatrix} \mapsto \widehat{\Sigma}_0 \begin{bmatrix} H_{\Delta}^{2\perp} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad U_{\widehat{\sigma}_0, \widehat{\Sigma}_0} : \begin{bmatrix} \widehat{s}_2 H_{\Delta}^2 \\ 0 \end{bmatrix} \mapsto \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ H_{\Delta^*}^2 \\ 0 \\ 0 \end{bmatrix}.$$

Hence (10.19) transforms to the equivalent condition in  $\mathcal{L}^{\widehat{\Sigma}_0}$ :

$$\begin{aligned} \text{clos } \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ \mathcal{K}''_-[t] \\ \mathcal{K}'_+ \end{bmatrix} &= \mathcal{L}^{\widehat{\Sigma}_0} \ominus \widehat{\Sigma}_0 \begin{bmatrix} H_{\Delta}^{2\perp} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\ \text{clos } \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ \mathcal{K}''_- \\ \mathcal{K}'_+[t^{-1}] \end{bmatrix} &= \mathcal{L}^{\widehat{\Sigma}_0} \ominus \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ H_{\Delta^*}^2 \\ 0 \\ 0 \end{bmatrix}. \end{aligned} \quad (10.20)$$

Conditions (10.20) are just the functional-model equivalent of conditions (8.8).

It is easily checked that condition (10.18) is just the translation of (8.9) to this functional-model setting.

Conditions (8.10) for this case can be read off from the analyticity of  $t^{-1}\widehat{s}_2$ ,  $\widehat{s}_1$  and  $\widehat{s}$ , respectively. Similarly, conditions (8.11) can be read off from the zero appearing in the (1, 3) and (2, 4) entries of  $\widehat{\Sigma}_0$ , respectively. Condition (10.17) can be seen to be equivalent to (8.24). The second version of Theorem 8.4 applies to lead us to the desired result.  $\square$

Combining Theorem 10.3 with Theorem 10.1 leads to the following corollary. A result of this type was obtained by Adamjan-Arov-Krein in the context of the Nehari problem in [3], see also [42].

**Corollary 10.4.** *Suppose that we are given the data set*

$$(\mathcal{U}', \mathcal{K}'), \quad (\mathcal{U}'', \mathcal{K}''), \quad \mathcal{K}'_+ \subset \mathcal{K}', \quad \mathcal{K}''_- \subset \mathcal{K}''$$

*for a Lifting Problem, that we set*

$$\sigma'(dt) = (i_{\mathcal{K}'_+ \rightarrow \mathcal{K}'} )^* E_{\mathcal{U}'}(dt) i_{\mathcal{K}'_+ \rightarrow \mathcal{K}'}, \quad \sigma''(dt) = (i_{\mathcal{K}''_- \rightarrow \mathcal{K}''} )^* E_{\mathcal{U}''}(dt) i_{\mathcal{K}''_- \rightarrow \mathcal{K}''}$$

*and that we let  $\hat{s}_0$  be an  $\mathcal{L}(\mathcal{K}'_+, \mathcal{K}''_-)$ -valued measure so that*

$$\sigma_0 := \begin{bmatrix} \sigma'' & \hat{s}_0 \\ \hat{s}_0^* & \sigma' \end{bmatrix} \geq 0.$$

*Construct measures  $\hat{s}_1$  and  $\hat{s}_2$  as maximal factorable minorants as in (10.5)–(10.9) in Theorem 10.1 and define  $\hat{s}$  as in (9.10). Then  $\hat{s}_0$  is the central-measure symbol for the Lifting Problem associated with the operator  $X = \hat{s}_0(\mathbb{T}) \in \mathcal{L}(\mathcal{K}'_+, \mathcal{K}''_-)$  if and only if  $\hat{s}$  is analytic,  $\hat{s}(\mathbb{T}) = 0$  and condition (10.18) holds. In this case  $\hat{\Sigma}^0$  as in (9.3) is the associated central characteristic measure.*

## 11. THE CLASSICAL NEHARI PROBLEM

In this section we use the classical Nehari problem as an illustration of our results for the general Lifting Problem. The classical Nehari problem (time-domain version) can be stated as follows: *given a sequence  $\{\gamma_n\}_{n=-1,-2,\dots}$  of complex numbers indexed by the negative integers, characterize all continuations  $\{\gamma_n\}_{n=0,1,2,\dots}$  so that the bi-infinite Toeplitz matrix  $\Gamma_e = [\gamma_{i-j}]_{i,j \in \mathbb{Z}}$  has  $\|\Gamma_e\| \leq 1$  as an operator on  $\ell^2(\mathbb{Z})$ .* This can be put in the form of a Lifting Problem with data set

$$\mathcal{K}' = \mathcal{K}'' = \ell^2(\mathbb{Z}), \quad \mathcal{U}' = \mathcal{U}'' = J \text{ (the bilateral shift)}, \quad \mathcal{K}'_+ = \ell^2(\mathbb{Z}_+), \quad \mathcal{K}''_- = \ell^2(\mathbb{Z}_-), \\ X = [\gamma_{i-j}]_{i \in \mathbb{Z}_-, j \in \mathbb{Z}_+} : \ell^2(\mathbb{Z}_+) \rightarrow \ell^2(\mathbb{Z}_-).$$

The equivalent frequency-domain version is: *given the complex sequence  $\{\gamma_n\}_{n \in \mathbb{Z}_-}$ , characterize all  $L^\infty$ -functions  $\varphi$  on the unit circle  $\mathbb{T}$  with Fourier series representation  $\varphi(t) = \sum_{n=-\infty}^{\infty} \varphi_n t^n$  so that*

$$\|\varphi\|_\infty \leq 1 \text{ and } \varphi_n = \gamma_n \text{ for } n = -1, -2, -3, \dots$$

If we follow the conventions of Remark 7.3, then the symbol  $\{w_Y\}_{n \in \mathbb{Z}}$  for a solution  $Y$  of the time-domain problem is connected with the corresponding solution  $\varphi$  of the frequency-domain problem according to the formula

$$\varphi(t) = \sum_{n=-\infty}^{\infty} w_Y(n+1) t^n,$$

i.e., if we let  $\hat{w}_Y(t) = \sum_{n=-\infty}^{\infty} w_Y(n) t^n$ , then  $\varphi(t) = t^{-1} \hat{w}_Y(t)$ . The Redheffer parametrization for the set of all solutions of the frequency-domain problem then has the form

$$\varphi(t) = t^{-1} s_0(t) + t^{-1} s_2(t) (1 - \omega(t) s(t))^{-1} \omega(t) s_1(t)$$

for a free-parameter Schur-class function  $\omega$ , where  $s_0, s_1, s_2, s$  are as in (7.6). If we set

$$\tilde{s}_0(t) = t^{-1} s_0(t), \quad \tilde{s}_2(t) = t^{-1} s_2(t), \quad \tilde{s}(t) = s(t), \quad \tilde{s}_1(t) = s_1(t), \quad (11.1)$$

then the formula

$$\varphi(t) = \tilde{s}_0(t) + \tilde{s}_2(t) (1 - \omega(t) \tilde{s}(t))^{-1} \omega(t) \tilde{s}_1(t) \quad (11.2)$$



becomes the parametrization of the set of all solutions of the frequency-domain classical Nehari problem associated with the sequence  $\{\gamma_n = [\tilde{s}_0]_n\}_{n=-1,-2,\dots}$  where  $\tilde{s}_0(t) = \sum_{n=-\infty}^{\infty} [\tilde{s}_0]_n t^n$  is the *central solution*. If we model  $\mathcal{K}'$  and  $\mathcal{K}''$  as  $L^2$  with both scale operators given by  $c \in \mathbb{C} \mapsto c \in L^2$ , then the central characteristic measure given by (9.2) becomes a multiple of Lebesgue measure:

$$\hat{\Sigma}_0 = \begin{bmatrix} 1 & \tilde{s} & 0 & \tilde{s}_1 \\ \tilde{s}^* & 1 & \tilde{s}_2^* & 0 \\ 0 & \tilde{s}_2 & 1 & \tilde{s}_0 \\ \tilde{s}_1^* & 0 & \tilde{s}_0^* & 1 \end{bmatrix} \cdot m.$$

In this section we simplify notation and write simply  $\hat{\Sigma}_0$  for the density against Lebesgue measure (i.e., we drop the  $\cdot m$  factor).

With all these conventions in order, the inverse problem can be formulated simply as: *characterize which  $2 \times 2$   $L^\infty$ -matrices  $\begin{bmatrix} \tilde{s} & \tilde{s}_1 \\ \tilde{s}_2 & \tilde{s}_0 \end{bmatrix}$  arise as the Redheffer coefficient matrix for a frequency-domain classical Nehari problem.* Obviously a necessary condition is that  $\|\tilde{s}_0\|_\infty \leq 1$ . In case  $\log(1 - |\tilde{s}_0(t)|^2)$  is not integrable with respect to Lebesgue measure over  $\mathbb{T}$ , then the solution of the associated Nehari problem is unique; hence, for  $\tilde{s}_0$  to be the central solution for an indeterminate problem, it is also necessary that  $\log(1 - |\tilde{s}_0(t)|^2)$  be integrable. In this case, it is well known (see e.g. [28]) that there is a unique outer function  $a$  with  $a(0) > 0$  so that

$$|a(t)|^2 = 1 - |\tilde{s}_0(t)|^2 \text{ a.e. for } t \in \mathbb{T}. \quad (11.3)$$

From Theorem 10.1 (with the appropriate adjustments caused by (11.1)), we see that then necessarily  $\tilde{s}_0$  determines  $\tilde{s}_1$  and  $\tilde{s}_2$  uniquely up to unimodular constant factors

$$\tilde{s}_1 = \tilde{s}_2 = a. \quad (11.4)$$

According to formula (10.16) in Theorem 10.3,  $\tilde{s}$  is then essentially uniquely determined by the condition

$$\begin{aligned} \begin{bmatrix} 1 & \tilde{s} \\ \tilde{s}^* & 1 \end{bmatrix} &= \begin{bmatrix} 0 & a \\ \bar{a} & 0 \end{bmatrix} \begin{bmatrix} 1 & \tilde{s}_0 \\ \tilde{s}_0^* & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & a \\ \bar{a} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a \\ \bar{a} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\bar{a}a} & -\frac{\tilde{s}_0^*}{\bar{a}a} \\ -\frac{\tilde{s}_0}{\bar{a}a} & \frac{1}{\bar{a}a} \end{bmatrix} \begin{bmatrix} 0 & a \\ \bar{a} & 0 \end{bmatrix} \text{ (where we use } 1 - |\tilde{s}_0|^2 = \bar{a}a \text{)} \\ &= \begin{bmatrix} 1 & -\frac{a}{\bar{a}}\tilde{s}_0^* \\ -\frac{\bar{a}}{a}\tilde{s}_0 & 1 \end{bmatrix} \end{aligned}$$

from which we read off that necessarily

$$\tilde{s} = -\frac{a}{\bar{a}}\tilde{s}_0^* =: b. \quad (11.5)$$

For our setting here (with coefficient spaces chosen to be  $\mathbb{C}$  rather than the whole spaces  $\mathcal{K}'_+ = \ell^2(\mathbb{Z}_+)$  and  $\mathcal{K}''_- = \ell^2(\mathbb{Z}_-)$ ), condition (10.17) translates simply to

$$\tilde{s}(0) = 0. \quad (11.6)$$

A pair  $(a, b)$  that satisfies the above properties, namely

- (1)  $a, b \in H^\infty$ ,
- (2)  $a$  is outer,  $b(0) = 0$ , and
- (3)  $|a|^2 + |b|^2 = 1$  almost everywhere,

is called a  $\gamma$ -generating pair in the sense that any function  $\varphi$  of the form (11.2) with  $\tilde{s}, \tilde{s}_1, \tilde{s}_2, \tilde{s}_0$  given by

$$\tilde{S} = \begin{bmatrix} \tilde{s} & \tilde{s}_1 \\ \tilde{s}_2 & \tilde{s}_0 \end{bmatrix} = \begin{bmatrix} b & a \\ a & -\frac{a}{\bar{a}}\bar{b} \end{bmatrix} \quad (11.7)$$

has  $\|\varphi\|_\infty \leq 1$  with negative Fourier coefficients  $[\varphi]_{-1}, [\varphi]_{-2}, \dots$  independent of the choice of free-parameter Schur-class function  $\omega$ .

If it is the case that the formula (11.2) parametrizes the set of *all* solutions of a Nehari problem, it is then said that  $(a, b)$  is a *Nehari pair*. It is known that not every  $\gamma$ -generating pair is a Nehari pair—see [39, 40, 41] for background on this and for some more refined results. The following characterization of Nehari pairs was obtained in [40, 42]. We now show how the result can be obtained as a corollary of Theorem 9.3.

**Theorem 11.1.** [40, 42] *A  $\gamma$ -generating pair  $(a, b)$  is a Nehari pair if and only if (in addition to conditions (1) – (3) above)  $(a, b)$  satisfies the fourth condition*

$$\begin{bmatrix} \bar{b} \\ P_{H^2} \bar{a} \end{bmatrix} \in \text{clos} \left\{ \begin{bmatrix} \bar{a}h \\ P_{H^2} \tilde{s}_0 h \end{bmatrix} : h \in H^{2\perp} \right\} \quad (11.8)$$

where  $\tilde{s}_0 := -\frac{a}{\bar{a}}\bar{b}$ .

*Proof.* By Theorem 9.3 above, properties (9.8) complete the list of necessary and sufficient conditions for a given  $\gamma$ -generating pair  $(a, b)$  to be a Nehari pair. When specialized to the Nehari problem, they read as follows: a  $\gamma$ -generating pair  $(a, b)$  is a Nehari pair if and only if

$$\begin{bmatrix} 1 \\ \bar{b} \\ 0 \\ \bar{a} \end{bmatrix} \mathbb{C} = \text{clos} \begin{bmatrix} 1 & b & 0 & a \\ \bar{b} & 1 & \bar{a} & 0 \\ 0 & a & 1 & \tilde{s}_0 \\ \bar{a} & 0 & \tilde{s}_0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ H^{2\perp} \\ H^2 \end{bmatrix} \ominus \text{clos} \begin{bmatrix} 1 & b & 0 & a \\ \bar{b} & 1 & \bar{a} & 0 \\ 0 & a & 1 & \tilde{s}_0 \\ \bar{a} & 0 & \tilde{s}_0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ H^{2\perp} \\ tH^2 \end{bmatrix} \quad (11.9)$$

and

$$\bar{t} \begin{bmatrix} b \\ 1 \\ a \\ 0 \end{bmatrix} \mathbb{C} = \text{clos} \begin{bmatrix} 1 & b & 0 & a \\ \bar{b} & 1 & \bar{a} & 0 \\ 0 & a & 1 & \tilde{s}_0 \\ \bar{a} & 0 & \tilde{s}_0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ H^{2\perp} \\ H^2 \end{bmatrix} \ominus \text{clos} \begin{bmatrix} 1 & b & 0 & a \\ \bar{b} & 1 & \bar{a} & 0 \\ 0 & a & 1 & \tilde{s}_0 \\ \bar{a} & 0 & \tilde{s}_0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \bar{t}H^{2\perp} \\ H^2 \end{bmatrix}. \quad (11.10)$$

Since for every  $\gamma$ -generating pair both sides of (11.9) are of dimension one and

$$\begin{bmatrix} 1 \\ \bar{b} \\ 0 \\ \bar{a} \end{bmatrix} \perp \begin{bmatrix} 1 & b & 0 & a \\ \bar{b} & 1 & \bar{a} & 0 \\ 0 & a & 1 & \tilde{s}_0 \\ \bar{a} & 0 & \tilde{s}_0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ H^{2\perp} \\ tH^2 \end{bmatrix},$$

then (for  $\gamma$ -generating pairs) (11.9) is equivalent to

$$\begin{bmatrix} 1 \\ \bar{b} \\ 0 \\ \bar{a} \end{bmatrix} \in \text{clos} \begin{bmatrix} 1 & b & 0 & a \\ \bar{b} & 1 & \bar{a} & 0 \\ 0 & a & 1 & \tilde{s}_0 \\ \bar{a} & 0 & \tilde{s}_0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ H^{2\perp} \\ H^2 \end{bmatrix}. \quad (11.11)$$

By a similar argument we see that (11.10) is equivalent to

$$\bar{t} \begin{bmatrix} b \\ 1 \\ a \\ 0 \end{bmatrix} \in \text{clos} \begin{bmatrix} 1 & b & 0 & a \\ \bar{b} & 1 & \bar{a} & 0 \\ 0 & a & 1 & \tilde{s}_0 \\ \bar{a} & 0 & \tilde{s}_0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ H^{2\perp} \\ H^2 \end{bmatrix}. \quad (11.12)$$

We first note some general principles concerning the space  $\mathcal{L}^{\widehat{\Sigma}_0}$ . Since  $\widetilde{S} = \begin{bmatrix} b & a \\ a & \tilde{s}_0 \end{bmatrix}(t)$  is unitary for almost all  $t \in \mathbb{T}$ , it can be seen as a consequence of Theorem 3.7 and some Schur-complement computations that  $f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$  is in  $\mathcal{L}^{\widehat{\Sigma}_0}$  if and only if

$$f \in L^2 \text{ and } \begin{bmatrix} f_1 \\ f_3 \end{bmatrix} = \widetilde{S} \begin{bmatrix} f_2 \\ f_4 \end{bmatrix}. \quad (11.13)$$

This, in particular, implies that for every  $f \in \mathcal{L}^{\widehat{\Sigma}_0}$  we have

$$f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \widehat{\Sigma}_0 \begin{bmatrix} 0 \\ f_2 \\ 0 \\ f_4 \end{bmatrix} = \widehat{\Sigma}_0 \begin{bmatrix} f_1 \\ 0 \\ f_3 \\ 0 \end{bmatrix}.$$

It then follows that

$$\|f\|_{\mathcal{L}^{\widehat{\Sigma}_0}} = \left\| \begin{bmatrix} f_2 \\ f_4 \end{bmatrix} \right\|_{L^2} \text{ and } \|f\|_{\mathcal{L}^{\widehat{\Sigma}_0}} = \left\| \begin{bmatrix} f_1 \\ f_3 \end{bmatrix} \right\|_{L^2}. \quad (11.14)$$

From the first equality in (11.14) we see that (11.11) is equivalent to

$$\begin{bmatrix} \bar{b} \\ \bar{a} \end{bmatrix} \in \text{clos} \begin{bmatrix} \bar{a} & 0 \\ \tilde{s}_0 & 1 \end{bmatrix} \begin{bmatrix} H^{2\perp} \\ H^2 \end{bmatrix}. \quad (11.15)$$

where now this closure and all closures to follow are computed in the  $L^2$ -metric. The latter condition is equivalent to

$$\begin{bmatrix} \bar{b} \\ \bar{a} - \frac{\bar{b}}{a(0)} \end{bmatrix} \in \text{clos } P_{H^{2\perp}} \begin{bmatrix} \bar{a} \\ \tilde{s}_0 \end{bmatrix} H^{2\perp}. \quad (11.16)$$

From the second equality in (11.14) we see that (11.11) is also equivalent to

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \text{clos} \begin{bmatrix} 0 & a \\ 1 & \tilde{s}_0 \end{bmatrix} \begin{bmatrix} H^{2\perp} \\ H^2 \end{bmatrix}. \quad (11.17)$$

which in turn is equivalent to

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \text{clos } P_{H^2} \begin{bmatrix} a \\ \tilde{s}_0 \end{bmatrix} H^2. \quad (11.18)$$

Similarly, the first of equations (11.14) converts (11.10) to the equivalent form

$$\bar{t} \begin{bmatrix} b \\ a \end{bmatrix} \in \text{clos} \begin{bmatrix} 0 & a \\ 1 & \tilde{s}_0 \end{bmatrix} \begin{bmatrix} H^{2\perp} \\ H^2 \end{bmatrix}. \quad (11.19)$$

which in turn can be rewritten as

$$\begin{bmatrix} b/t \\ \frac{a - a(0)}{t} \end{bmatrix} \in \text{clos } P_{H^2} \begin{bmatrix} a \\ \tilde{s}_0 \end{bmatrix} H^2, \quad (11.20)$$

while the second of equations (11.14) can be used to convert (11.10) to the seemingly different equivalent form

$$\begin{bmatrix} \bar{t} \\ 0 \end{bmatrix} \in \text{Clos} \begin{bmatrix} \bar{a} & 0 \\ \bar{s}_0 & 1 \end{bmatrix} \begin{bmatrix} H^{2\perp} \\ H^2 \end{bmatrix}. \quad (11.21)$$

which in turn can be rewritten as

$$\begin{bmatrix} \bar{t} \\ 0 \end{bmatrix} \in \text{clos } P_{H^{2\perp}} \begin{bmatrix} \bar{a} \\ \bar{s}_0 \end{bmatrix} H^{2\perp}. \quad (11.22)$$

Next observe that (11.19) is just the complex-conjugate version of (11.15) and hence (11.19) and (11.15) are equivalent. We conclude that in fact (11.9) and (11.10) are equivalent to each other. Alternatively, to arrive at the same result, observe that (11.17) and (11.21) are complex-conjugate versions of each other, from which it follows that (11.9) and (11.10) are equivalent to each other.

Thus the  $\gamma$ -generating pair is a Nehari pair exactly when any one of the equivalent conditions (11.16), (11.18), (11.20), or (11.22) holds. In particular, we choose condition (11.16) to arrive at condition (11.8) in Theorem 11.1.  $\square$

**Remark 11.2.** It is known (see [66, Section 6]) that  $\gamma$ -generating pairs  $(a, b)$  are in one-to-one correspondence with extreme points of the unit ball of  $H^1$  via  $f = (a/(1-b))^2$  and via the same formula Nehari pairs are in one-to-one correspondence with exposed points for the unit ball of  $H^1$ . The above characterization of Nehari pairs is, at the same time, the best known characterization of exposed points.

**Remark 11.3.** We note also that (for  $\gamma$ -generating pairs) (11.9)/(11.10) is equivalent to

$$\mathcal{S} := \begin{bmatrix} 1 \\ \bar{b} \\ 0 \\ \bar{a} \end{bmatrix} H^{2\perp} \oplus \text{clos} \begin{bmatrix} 1 & b & 0 & a \\ \bar{b} & 1 & \bar{a} & 0 \\ 0 & a & 1 & \bar{s}_0 \\ \bar{a} & 0 & \bar{s}_0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ H^{2\perp} \\ H^2 \end{bmatrix} \oplus \begin{bmatrix} b \\ 1 \\ a \\ 0 \end{bmatrix} H^2 = L^{\widehat{\mathcal{S}}_0}, \quad (11.23)$$

which is (9.9). Indeed, if (11.9)/(11.10) hold true, then (as in Theorems 8.4 and 9.3)  $\mathcal{S}$  is a  $*$ -cyclic reducing subspace, therefore, it agrees with  $L^{\widehat{\mathcal{S}}_0}$ . Conversely, assume that equality in (11.23) holds. We check directly that

$$t \begin{bmatrix} 1 \\ \bar{b} \\ 0 \\ \bar{a} \end{bmatrix} \bar{t} = \begin{bmatrix} 1 \\ \bar{b} \\ 0 \\ \bar{a} \end{bmatrix} \perp \begin{bmatrix} 1 \\ \bar{b} \\ 0 \\ \bar{a} \end{bmatrix} H^{2\perp} \oplus \text{clos} \begin{bmatrix} 1 & b & 0 & a \\ \bar{b} & 1 & \bar{a} & 0 \\ 0 & a & 1 & \bar{s}_0 \\ \bar{a} & 0 & \bar{s}_0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ H^{2\perp} \\ tH^2 \end{bmatrix} \oplus \begin{bmatrix} b \\ 1 \\ a \\ 0 \end{bmatrix} H^2. \quad (11.24)$$

From the decomposition (11.23) we conclude that

$$\begin{bmatrix} 1 \\ \bar{b} \\ 0 \\ \bar{a} \end{bmatrix} \in \text{clos} \begin{bmatrix} 1 & b & 0 & a \\ \bar{b} & 1 & \bar{a} & 0 \\ 0 & a & 1 & \bar{s}_0 \\ \bar{a} & 0 & \bar{s}_0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ H^{2\perp} \\ H^2 \end{bmatrix}.$$

Combining this with (11.24) we see that

$$t \begin{bmatrix} 1 \\ \bar{b} \\ 0 \\ \bar{a} \end{bmatrix} \bar{t} = \begin{bmatrix} 1 \\ \bar{b} \\ 0 \\ \bar{a} \end{bmatrix} \in \text{clos} \begin{bmatrix} 1 & b & 0 & a \\ \bar{b} & 1 & \bar{a} & 0 \\ 0 & a & 1 & \bar{s}_0 \\ \bar{a} & 0 & \bar{s}_0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ H^{2\perp} \\ H^2 \end{bmatrix} \ominus \text{clos} \begin{bmatrix} 1 & b & 0 & a \\ \bar{b} & 1 & \bar{a} & 0 \\ 0 & a & 1 & \bar{s}_0 \\ \bar{a} & 0 & \bar{s}_0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ H^{2\perp} \\ tH^2 \end{bmatrix}.$$

This implies the equality in (11.9) since both sides there are of dimension one. By a similar argument we get the equality in (11.10).

The equality in (11.23) can be restated equivalently as follows:  $f \in \mathcal{L}^{\widehat{\Sigma}_0}$  is orthogonal to  $\mathcal{S}$ , if and only if  $f = 0$ . On the other hand,  $f \in \mathcal{L}^{\widehat{\Sigma}_0}$  is orthogonal to  $\mathcal{S}$ , if and only if  $f_1, f_3$  are in  $H^2$  and  $f_2, f_4$  are in  $H^{2\perp}$ . In view of (11.13) this means that equality in (11.23) is equivalent to the property that the equation

$$\widetilde{S}^* \begin{bmatrix} u_+ \\ v_+ \end{bmatrix} = \begin{bmatrix} u_- \\ v_- \end{bmatrix} \text{ with } \begin{bmatrix} u_+ \\ v_+ \end{bmatrix} \in H^2 \text{ and } \begin{bmatrix} u_- \\ v_- \end{bmatrix} \in H^{2\perp} \quad (11.25)$$

has only the trivial solution, where  $\widetilde{S}^*$  is as in (11.7). Thus, we get an alternative characterization of Nehari pairs (that also was obtained in [40, 42]):

**Theorem 11.4.** [40, 42] A  $\gamma$ -generating pair is a Nehari pair if and only if (11.25) has only the trivial solution.

**Remark 11.5.** In the context of the scalar Nehari problem, the parametrization formula (7.12) reads in our terms as

$$M_w|_{H^2} = \Gamma + \Phi_{22}\sqrt{I - \Gamma^*\Gamma} + \Phi_{21}\omega(I - \Phi_{11}\omega)^{-1}\Phi_{12}\sqrt{I - \Gamma^*\Gamma} \quad (11.26)$$

where  $\Gamma$  is the given Hankel operator from  $H^2$  to  $H^{2\perp}$ ,  $M_w|_{H^2}: H^2 \rightarrow L^2$  is the restriction to  $H^2$  of the multiplication operator  $M_w: f(t) \mapsto w(t)f(t)$  on  $L^2$  and  $w$  is a solution of the given Nehari problem, and  $\omega$  is the free-parameter Schur-class function (see (7.10) and (7.12)). The Redheffer coefficient matrix  $\Psi$  of [31] (see (7.11)) simplifies to

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix} : \begin{bmatrix} h_1 \\ \sqrt{I - \Gamma^*\Gamma}h_2 \end{bmatrix} \rightarrow \begin{bmatrix} \widetilde{s}h_1 + \widetilde{s}_1h_2 \\ \widetilde{s}_2h_1 + P_{H^2}\widetilde{s}_0h_2 \end{bmatrix}, \quad h_1, h_2 \in H^2; \quad (11.27)$$

$\widetilde{s}, \widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_0$  arise from a  $\gamma$ -generating pair  $(a, b)$  as in (11.7). From the unitary property of  $\begin{bmatrix} \widetilde{s} & \widetilde{s}_1 \\ \widetilde{s}_2 & \widetilde{s}_0 \end{bmatrix}$  as a multiplication operator on  $L^2 \oplus L^2$  and the fact that  $P_{H^{2\perp}}\widetilde{s}_0|_{H^2} = \Gamma$ , it follows that  $\Psi$  is isometric as an operator from  $\left[ \frac{H^2}{\text{Ran}(I - \Gamma^*\Gamma)} \right] = \left[ \frac{H^2}{H^2} \right]$  (the equality holds since the problem is indeterminate) to  $\left[ \frac{H^2}{H^2} \right]$ . Hence  $\Psi$  is also coisometric exactly when the set

$$\left\{ \begin{bmatrix} \widetilde{s}h_1 + \widetilde{s}_1h_2 \\ \widetilde{s}_2h_1 + P_{H^2}\widetilde{s}_0h_2 \end{bmatrix} : h_1, h_2 \in H^2 \right\} \quad (11.28)$$

is dense in  $H^2 \oplus H^2$ .

Note, that  $H^2$  in the first entry of the initial space for  $\Psi$  and  $H^2$  in both entries of the target space for  $\Psi$  should be understood as  $H^2(\mathbb{C})$  since  $\mathcal{D}_{\rho^*}$ ,  $\mathcal{G}$  and  $\mathcal{D}_{T'}$  are all of dimension one, while  $H^2$  in the second entry of the initial space for  $\Psi$  is  $\mathcal{D}_\Gamma$ .

Conversely, given any matrix  $\widetilde{S} = \begin{bmatrix} \widetilde{s} & \widetilde{s}_1 \\ \widetilde{s}_2 & \widetilde{s}_0 \end{bmatrix}$  arising from a  $\gamma$ -generating pair  $(a, b)$  as in (11.7). Define a Hankel operator as  $\Gamma = P_{H^{2\perp}}\widetilde{s}_0|_{H^2}$  and a matrix

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix} : H^2 \oplus H^2 \rightarrow H^2 \oplus H^2$$

as in (11.27). Then  $\Psi$  is an isometry. Assume that (11.28) is dense in  $H^2 \oplus H^2$ . Then  $\Psi$  is a coisometry. Hence, Theorem 0.3 of [31] applies to this  $\Psi$ . The theorem tells us that  $\Psi$  is the Redheffer coefficient matrix as in (7.12) for some (in general not unique) relaxed commutant lifting problem. The data of such a problem presented

in the proof of Theorem 0.3 in [31] is as follows:  $A : \mathbb{C} \oplus H^2 \rightarrow \mathbb{C}$  is a projection on the first entry,  $T' = 0$  on  $\mathbb{C}$ ,  $R$  is a certain contraction from a certain subspace  $\mathcal{F} \subset H^2$  to  $\mathbb{C} \oplus H^2$  and  $Q$  is an embedding of  $\mathcal{F}$  into  $\mathbb{C} \oplus H^2$ .

However, the density property of (11.28) can be equivalently formulated as: *the equation*

$$\tilde{S}^* \begin{bmatrix} u_+ \\ v_+ \end{bmatrix} = \begin{bmatrix} u_- \\ v_- \end{bmatrix}, \quad \begin{bmatrix} u_+ \\ v_+ \end{bmatrix} \in H^2, \begin{bmatrix} u_- \\ v_- \end{bmatrix} \in H^{2\perp}$$

*has only the trivial solution.* In view of Theorem 11.4 above, the latter is equivalent to the property that  $\tilde{S}$  is the Redheffer coefficient matrix for a Nehari problem. We conclude that: *the operator  $\Psi$  (11.27) is coisometric (and hence unitary) if and only if  $\begin{bmatrix} \tilde{s} & \tilde{s}_1 \\ \tilde{s}_2 & \tilde{s}_0 \end{bmatrix}$  is the Redheffer coefficient matrix (in the sense of the present paper) for a Nehari problem.* Thus, in this case the RCL problem can be taken to have the special form of a Nehari problem. In this way we arrive at an improved version (in the context of the Nehari problem) of the result from [31].

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